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ORIGINAL RESEARCH

Characterization of Non-deterministic Chaos in Two-dimensional Non-smooth Vector Fields

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Abstract
Our context is Filippov systems defined on two-dimensional manifolds having a finite number of tangency points. We prove that topological transitivity is a necessary and sufficient condition for the occurrence of non-deterministic chaos when the Filippov system has non-empty sliding or escaping regions. A fundamental result for continuous flows is the equivalence of topological transitivity and existence of a dense orbit. We prove in our setting that topological transitivity for Filippov systems is indeed equivalent to the existence of a dense Filippov orbit, although, in contrast to the continuous case, we are not able to guarantee that the dense orbit implies the existence of a residual set of dense orbits. Finally, we prove that, in this context, topological transitivity implies strictly positive topological entropy for the Filippov system. This calculation is made using techniques similar to those from symbolic dynamics.

Keywords Piecewise smooth differential system · Filippov system · Transitivity · Topological entropy

Mathematics Subject Classification 34A36 · 34A60 · 37G15 · 37B40

Introduction

Ordinary differential equations (ODEs) are well-established tools for modeling a wide range of real-world problems. Recently, however, the consideration of discontinuities in ODE solutions has gained prominence in the study of dynamical systems. Discontinuous ODEs appear in various applications, such as control theory, the dynamics of bouncing

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balls, predator foraging, anti-lock braking systems (ABS), and mechanical systems with component collisions, friction, sliding, or squealing. For detailed discussions on these applications, refer to [4, 9–11, 15, 18–20, 22] and related references.

One prominent approach to discontinuous ODEs is Filippov's theory (see [14]), which allows solutions to "slide" through discontinuity regions according to predefined rules. This method can lead to non-unique solutions and introduce unusual behaviors, even in familiar scenarios such as planar phase portraits. Alternative approaches to discontinuous ODEs are well-articulated in Cortés [7]. In this paper, we focus on ODEs with discontinuities, referred to as *Filippov systems*, in accordance with Filippov's convention.

We investigate the chaotic behavior of Filippov systems defined on two-dimensional manifolds. In the realm of smooth deterministic dynamical systems, chaos is traditionally defined by Devaney as a system that is topologically transitive, sensitive to initial conditions, and has a dense set of periodic orbits. In [2], the authors refine these conditions to show that the topological transitivity and density of periodic solutions are sufficient for chaos. We extend this concept to Filippov systems, where the notion of chaos is adjusted to account for the non-deterministic nature of forward orbits. For further insights into non-deterministic chaos, see [5, 6, 12], and [21]. The definition of *non-deterministic chaos* is detailed in Sect. 3.

Our paper focuses on a two-dimensional manifold M with a Filippov system that has a finite number of tangency points. We demonstrate that a Filippov system is topologically transitive if and only if it has a dense Filippov orbit (Theorem 2.1). Furthermore, we show in Theorem 2.2 that a transitive Filippov system with non-empty sliding or escaping regions exhibits non-deterministic chaos (i.e., has a dense orbit, dense periodic orbits, and sensitivity to initial conditions). Thus, we establish that **transitivity is equivalent to non-deterministic chaos**. Additionally, we prove that in our context, transitivity also implies positive entropy (Theorem 2.3), which is often used to gauge the level of chaos in a system. Positive entropy signifies the exponential growth rate of distinct orbits, indicating a system's complexity and potential chaotic behavior.

This paper is organized as follows: Sect. 2 states the main results of the paper. In Sect. 3, we present the first concepts and definitions of Filippov systems which are going to be used throughout the paper. In Sect. 4, we prove Theorem 2.2; in Sect. 5, Theorem 2.1; and Theorem 2.3 is done in Sect. 6.

Main Results

In the classical context of continuous flows, topological transitivity is equivalent to the existence of a dense orbit (which is sometimes simply called transitivity). In fact, it is further equivalent to the existence of a residual set of transitive points, that is, points through which a dense orbit passes. In order to provide the theory of Filippov systems with a solid foundation, we address topological transitivity for Filippov systems in this paper.

We obtain that the same equivalence of topological transitivity and transitivity holds for a specific setting on two-dimensional manifolds, but due to the complexity of piecewise smooth systems, it is not as straightforward to obtain a residual set of transitive points.

In our framework, we begin by defining sets E_+ and E_- (refer to Definition 3.3 for their precise definitions, and to Subsection 3.1 for the definitions of stable and unstable sliding regions). The set E_+ consists of the regular points of the Filippov system that first lose uniqueness at the stable sliding region Σ^{ss} when evolving forward in time.

Specifically, it is the set of regular points p such that the regular orbit segment starting at p intersects Σ^{ss} at some positive time and does not intersect the closure of Σ^{s} prior to this. Similarly, the set E_{-} is defined as the set of regular points that first lose uniqueness at the unstable sliding (escaping) region Σ^{us} when evolving backward in time.

In the existing examples of transitive bidimensional Filippov systems found in the literature, the bean model and the sphere model [1, 5, 13], it is straightforward to verify that the sets E_{+} and E_{-} are dense. Consequently, assuming the density of these sets appears to be a natural hypothesis. However, we propose a weaker assumption: E_{+} is required to be dense only when $\Sigma^{ss} \neq \emptyset$, and E_{-} is assumed to be dense only when $\Sigma^{us} \neq \emptyset$.

Theorem 2.1 *Let M be a two-dimensional manifold with a Filippov system having a finite number of tangency points. Assume that either $\Sigma^{ss} \neq \emptyset$ and E_{+} is dense, or that $\Sigma^{us} \neq \emptyset$ and E_{-} is dense. Then the Filippov system is topologically transitive on M if, and only if, there exists a dense Filippov orbit.*

Since for continuous flows topological transitivity is equivalent to the existence of a residual set of dense orbit, one good question that we are not able to prove so far is:

Question 1 *In the context of Theorem 2.1 does the existence of a dense orbit imply the existence of a residual set with dense orbit?*

The following result is proved before proceeding to the proof of Theorem 2.1. We also assume the hypotheses about E_{+} or E_{-} as in the preceding theorem.

Theorem 2.2 *Let M be a two-dimensional manifold and Z a transitive Filippov system. Assume that either $\Sigma^{ss} \neq \emptyset$ and E_{+} is dense, or that $\Sigma^{us} \neq \emptyset$ and E_{-} is dense. Then the following statements hold:*

1. *There is a dense set Δ such that, for every $x \in \Delta$,*

1.1 *there is a dense orbit through x;*
1.2 *the periodic orbits through x form a dense set;*

2. *The Filippov system has sensitive dependence on initial conditions.*

As in Question 1, is it possible to obtain such a set Δ from Theorem 2.2 that is residual? After the proof of Theorem 2.2 on Subsection 4.1 we make a discussion concerning the issues to get a residual set.

Our last result shows that topological transitivity (or transitivity) implies positive entropy, which may be seen as well as a measure of chaos.

Theorem 2.3 *Assume that M is a two-dimensional manifold with a transitive Filippov system. If the sliding or escaping regions are non-empty, then the Filippov system has positive topological entropy.*

Preliminaries

Filippov Systems

Let M be a 2-dimensional C^k closed Riemannian manifold and $\Sigma \subset M$ a set formed by the union of n smooth curves Σ_i which are pairwise disjoint, where $\Sigma_i = h_i^{-1}(0)$ with $i = 1, \ldots, n$ and $h_i : M \longrightarrow \mathbb{R}$ is a smooth function having 0 as regular value. We write $\Sigma = \bigcup_{i=1}^{n} \Sigma_i$ and assume that Σ splits M into $n+1$ disjoint **regular regions** R_i, on which we define $n+1$ smooth vector fields X_i, $i \in \{1, \ldots, n+1\}$. We call Σ the **switching manifold** and assume that it is contained in the boundary of the regions R_i. Around each curve Σ_i there are two of the $n+1$ regions R_i, which we denote R_i^+ (where h_i is positive) and R_i^- (where h_i is negative), and their respective vector fields are denoted X_i^+ and X_i^-. We denote by $\langle v, v' \rangle_x$ the Riemannian metric on M at a point $x \in M$ on vectors v and v', and by $d : M \times M \to \mathbb{R}$ the distance function induced by the Riemannian metric.

We call $\mathfrak{X}^r$ the space of C^r-vector fields $X : M \longrightarrow TM$ with $1 \leq r \leq k$ where k is sufficiently large. We denote the **Lie derivative** of a function h relative to a vector field X at a point x by $Xh(x) := \langle \nabla f(x), X(x) \rangle_x$.

There are different types of points on Σ. They can be classified as follows:

1. At **crossing points**, both vector fields X_i^+ and X_i^- point to the same side of Σ_i. We define the crossing region as

$$\Sigma_i^{\mathrm{c}} := \left\{ x \in \Sigma_i \mid X_i^+ h_i(x) X_i^- h_i(x) > 0 \right\}.$$

2. At **sliding points**, the vectors fields X_i^+ and X_i^- point to opposite sides of Σ_i. We define the sliding region as

$$\Sigma_i^{\mathrm{s}} := \left\{ x \in \Sigma_i \mid X_i^+ h_i(x) X_i^- h_i(x) < 0 \right\}.$$

The sliding region is subdivided in two regions:

2.1 At (**stable**) **sliding points**, the vectors fields X_i^+ and X_i^- point towards Σ_i, such that trajectories reach the sliding region in finite future time. We define the stable sliding region as

$$\Sigma_i^{\mathrm{ss}} := \left\{ x \in \Sigma_i^{\mathrm{s}} \mid X_i^+ h_i(x) < 0 \right\}.$$

2.2 At **escaping points** (or **unstable sliding points**), the vectors fields X_i^+ and X_i^- point away from Σ_i, such that trajectories reach the escaping region in finite past time. We define the escaping region as

$$\Sigma_i^{\mathrm{us}} := \left\{ x \in \Sigma_i^{\mathrm{s}} \mid X_i^+ h_i(x) > 0 \right\}.$$

3. At **tangency points**, one of the vector fields X_i^+ and X_i^- is tangent to Σ_i and the other may be tangent or not. In this first case, we refer to it as a **regular tangency** point and, in the second case, a **double tangency** point. We define the tangent region as

$$\Sigma_i^{\mathrm{t}} := \left\{ x \in \Sigma_i \mid X_i^+ h_i(x) X_i^- h_i(x) = 0 \right\}.$$

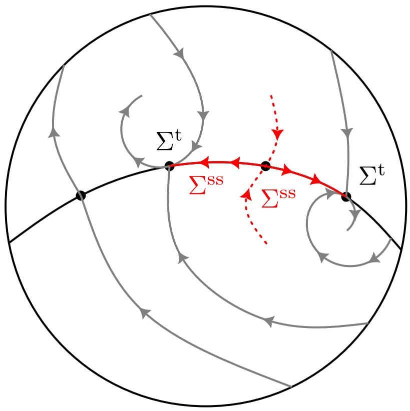

Fig. 1 A Filippov system on $\mathbb{S}^2$. The continuous path between the tangency points is a stable sliding region Σ^{ss} (notice the pseudo-equilibrium reached by the two dashed orbits)

The union over $i \in \{1, \dots, n\}$ of the above sets are denoted, respectively, by Σ^c, Σ^s, Σ^{ss}, Σ^{us} and Σ^t. Some of these points are represented in Fig. 1.

Call Ω the space of **piecewise smooth vector fields** $Z : M \to TM$ such that

$$Z(x) = \begin{cases} X_i(x), & \text{if } x \in R_i \\ \dfrac{X_i^- h_i(x) X_i^+ - X_i^+ h_i(x) X_i^-}{X_i^- h_i(x) - X_i^+ h_i(x)}, & \text{if } x \in \Sigma_i^s \end{cases} \tag{1}$$

The vector field Z is not defined on $\Sigma^t \cup \Sigma^c$, as its value on these regions does not matter to the definition of a Filippov orbit (Definition 3.1). In Eq. 1 the field is defined on Σ^s using *Filippov's convention*: it consists of the unique vector that is both tangent to Σ_i and is a

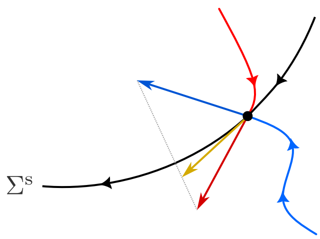

Fig. 2 Defining a vector field on the switching manifold Σ. The vector tangent to Σ is a convex combination of the other two vectors

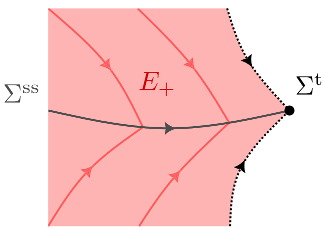

Fig. 3 In the figure, all the points in the red region belong to the set E_+, as they are regular points that reach Σ^{ss} before reaching any other point in $\Sigma^s \cup \Sigma^t$. The points in the dotted orbits that reach the tangency point in Σ^t do not belong to E_+

convex combination of the vector fields X_i^+ and X_i^- (see Fig. 2). This is called the **sliding vector field**.

In this work we assume there are finite tangency points and also that there are no **pseudo-equilibrium points**,[1] that is, points $x \in \Sigma^s$ where $Z(x) = 0$.

The following definition can be found in [17].

Definition 3.1 A **Filippov orbit** (or **solution**) of Z is a map $\gamma : \mathbb{R} \to M$ such that: if $\gamma(t)$ is outside Σ, then the orbit is locally determined by the smooth vector fields X_i; on Σ^s, it is determined by the (sliding) vector field on Σ^s (Eq. 1), considering that, in the case the orbit goes through an escaping region, it may exit at any arbitrary moment; on Σ^c, the orbit is a concatenation of the orbits on the regular region of each side; and on Σ^t, the orbit is an extension of the regular or sliding orbits.

Definition 3.2 The **saturation** of a set $A \subset M$, denoted A_φ, is the union of every Filippov orbit with initial condition on A.

For our purposes we will mainly deal with the set Σ^s_φ. Now we define the sets E_+ and E_- (see Fig. 3) that are present in the hypotheses of Theorems 2.2 and 2.1.

Definition 3.3 We define the following sets:

1. E_+ is the set of points $p \in M \setminus \Sigma$ that first lose uniqueness going forward in Σ^{ss} (that is, there is an orbit γ starting at p and $t_s > 0$ such that $\gamma(t_s) \in \Sigma^{ss}$ and, for every $0 < t < t_s$, $\gamma(t)$ is in a regular region R_i or crossing region).
2. E_- is the set of points of $M \setminus \Sigma$ that first lose uniqueness going backward in Σ^{us}.
3. D_+ is the set of points $p \in M \setminus \Sigma$ for which there is an orbit γ starting at p that reaches Σ^{ss} going forward (that is, there is $t_s > 0$ such that $\gamma(t_s) \in \Sigma^{ss}$).
4. D_- is the set of points $p \in M \setminus \Sigma$ for which there is an orbit starting at p that reaches Σ^{us} going backward.

Next, we introduce the definition of non-deterministic chaos. For simplicity, we will just say *chaotic Filippov systems*. This is the natural definition of chaos for Filippov systems when following the classical definition of Devaney (see [8]).

Definition 3.4 Let Z be a Filippov system on a manifold M (as in Eq. 1). We define the following:

1. Z is **topologically transitive** if given any two non-empty open sets U and V of M, there exists a Filippov orbit from a point of U to a point of V.
2. Z exhibits **sensitive dependence on initial conditions** if there exists a fixed $\delta > 0$ such that, for any non-empty open set U, there exist points $x, y \in U$, Filippov orbits γ_x and γ_y which start at x and y, respectively, and some time $t > 0$ such that $d(\gamma_x(t), \gamma_y(t)) > \delta$.
3. a Filippov orbit γ is **periodic** if there is $\tau \in \mathbb{R}$ such that $\gamma(t) = \gamma(t + \tau)$ for every $t \in \mathbb{R}$.
4. Z is **chaotic** if it is topologically transitive, has sensitive dependence on initial conditions and the union of all Filippov periodic orbits is a dense set.

[1] This is mainly for simplicity since we could only assume the pseudo-equilibrium points are separated and then consider in Lemma 4.2 only points of Σ^s that are not pseudo-equilibrium points.

Topology

We briefly state some definitions and notation related to topology. We denote the interior of a set A by $\mathrm{int}\,(A)$ and the complement of a set A by $A^{\complement} = M \setminus A$. A **residual** set is a set that is a countable intersection of sets whose interior is dense. This is equivalent to being the complement of a **meager** set, that is, a countable union of sets whose closure has empty interior. Residual sets represent a way to express the idea of "almost all" only using topology.

Orbit Spaces of Filippov Systems

In this subsection we follow definitions first presented in [1] and later expanded on in [16]. In the broad context of orbit spaces, we may take M to be any bounded connected Riemannian manifold and the Filippov field Z to be bounded (that is, its norm $\|Z\|$, given by the supremum, must be bounded) [16]. In our case here, we assume the particular context of the compact 2-dimensional Riemannian manifold M as exposed in the preceding Subsection 3.1.

The **orbit space** of the Filippov system Z over M is the set of all possible (Filippov) orbits of Z, denoted $\widetilde{M}$. Denoting by $d : M \times M \to \mathbb{R}$ the distance function induced by the Riemannian metric of M, we define the **supremum distance** $\widetilde{d}_{\sup}$ on the orbit space $\widetilde{M}$ by setting, for each $\gamma_0, \gamma_1 \in \widetilde{M}$,

$$\widetilde{d}_{\sup}(\gamma_0, \gamma_1) := \sum_{i \in \mathbb{Z}} \frac{1}{2^{|i|}} \sup_{i \leq t < i+1} d(\gamma_0(t), \gamma_1(t)).$$

The distance converges for every pair of orbits because the space is compact. This turns $\widetilde{M}$ into a metric space that is complete, separable and has no isolated points [16].

The dynamics on $\widetilde{M}$ is induced by Z: we define a flow $\widetilde{\Phi}^t$ (for each $t \in \mathbb{R}$) as

$$\begin{aligned} \widetilde{\Phi}^t : \widetilde{M} &\longrightarrow \widetilde{M} \\ \gamma &\longmapsto \begin{aligned} \widetilde{\Phi}^t(\gamma) : \mathbb{R} &\longrightarrow M \\ s &\longmapsto \gamma(t+s). \end{aligned} \end{aligned}$$

It can be shown that the flow $\widetilde{\Phi}$ is continuous [16].

On the orbit space, we have uniqueness of solutions, which is most often not true for Filippov systems, so we may use concepts and techniques of the general theory of flows on $\widetilde{M}$ to try and obtain information about Z. One example of this is done in the next subsection for the definition of topological entropy for Filippov systems: we define the topological entropy of Z as the topological entropy of the flow in the orbit space.

Topological Entropy

To define the topological entropy of a Filippov system, we follow the approach of [1]. We will first present the definition of topological entropy of Bowen-Dinaburg [3] for compact metric spaces. We will assume our space is compact. We will also use generating sets and will not define separated sets, but a complete presentation can be found in [23]. In this section we will denote generic compact metric spaces by K to avoid confusion with our previously defined manifold M.

Definition 3.5 Let (K, d) be a compact metric space, $f : K \to K$ a continuous transformation, $\varepsilon > 0$ and $n \in \mathbb{N}$. A (n, ε)**-generating set** is a set $E \subseteq K$ that satisfies the following: for every point $x \in K$, there is a point $a \in E$ such that $d(f^i(a), f^i(x)) < \varepsilon$ for every $i \in \{0, \dots, n-1\}$.

If we define the **dynamical distance** of order n (for the dynamics f) between $x, y \in K$ as

$$d_f^n(x, y) := \max_{0 \le i < n} d(f^i(x), f^i(y))$$

and the **dynamical ball** of center a, radius ε and order n (for the dynamics f) as

$$B_f^n(a, \varepsilon) := \left\{ x \in K \mid d_f^n(a, x) < \varepsilon \right\},$$

the preceding definition is equivalent to the condition $K \subseteq \bigcup_{a \in E} B_f^n(a, \varepsilon)$.

We define $g^n(f, \varepsilon)$ to be the **smallest cardinality** of a (n, ε)-generating set, that is

$$g^n(f, \varepsilon) := \min \left\{ \#E \mid E \subseteq K \subseteq \bigcup_{a \in E} B_f^n(a, \varepsilon) \right\}.$$

By compactness, this number is always finite.

The dynamical ball $B^n(a, \varepsilon)$ gives us information about which points stay ε-close to the center a for n units of time (i.e. n iterations of f), so the minimal cardinality $g^n(f, \varepsilon)$ counts how many different orbits of the system there are, but for an approximated time n and with an imprecision ε. We are interested in the exponential growth of the number of orbits of the system as time passes, so in the following definition we take the limit as $n \to \infty$ and $\varepsilon \to 0$ of the quantity $\frac{1}{n} \log g^n(f, \varepsilon)$. This limit can be shown [23] to be a well-defined number in $[0, \infty]$.

Definition 3.6 Let K be a compact metric space and $f : K \to K$ a continuous transformation. The **topological entropy of** f is

$$h(f) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log g^n(f, \varepsilon).$$

We can define the **topological entropy a continuous flow** Φ^t ($t \in \mathbb{R}$) in a compact metric space K in an analogous way using dynamical balls and generating sets, and its value is the same as the topological entropy of the time 1 map $\Phi^1 : K \to K$ of the flow [23], which is often simply taken as the definition of entropy for flows. This is the motivation for the definition of topological entropy for Filippov systems, first presented in [1], which we will use.

Definition 3.7 Let Z a Filippov system on a manifold M. The **topological entropy** of Z is the topological entropy of the time 1 map $\widetilde{\Phi}^1 : \widetilde{M} \to \widetilde{M}$ on the orbit space $\widetilde{M}$, denoted

$$h(Z) := h(\widetilde{\Phi}^1).$$

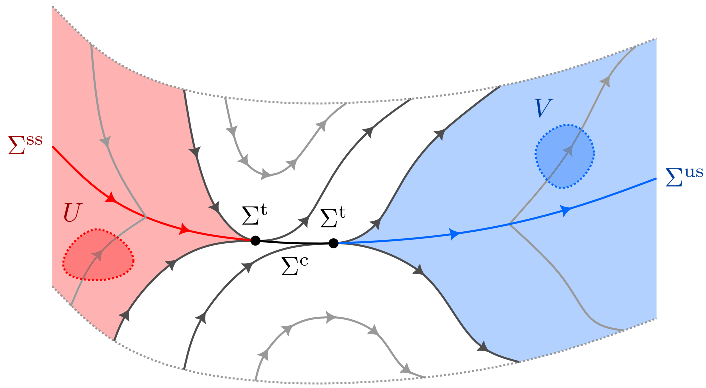

Fig. 4 An open set U that reaches Σ^{ss} flowing forward and an open set V that reaches Σ^{us} flowing backwards

Proof of Theorem 2.2

We shall prove a number of important lemmas that will be needed for the proof of our result. First we show how topological transitivity leads to the existence of a connection of points on the sliding region.

Lemma 4.1 *Assume topological transitivity. For every pair of points $q_0, q_1 \in \Sigma^s$, there is a Filippov orbit segment from q_0 to q_1.*

Proof The points q_0 and q_1 can be in either Σ^{ss} or Σ^{us}, so there are 4 possibilities of connections of q_0 and q_1 (see Fig. 5, which represents all 4 cases at once). We will describe how to connect them in all cases. If $q_0 \in \Sigma^{ss}$, since Σ^s is relatively open in Σ we can choose a point q_0' before q_0 (that is, q_0' is in the same connected component of Σ^{ss} as q_0 and it reaches q_0 through an orbit segment of the sliding vector field on Σ^{ss}) and an open set $U_0 \subset M$ around

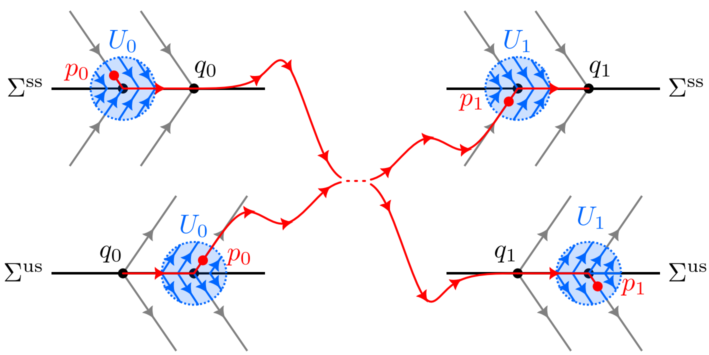

Fig. 5 The points q_0 and q_1 can each belong to either Σ^{ss} or Σ^{us}, so there are 4 cases to be considered. All 4 possibilities are represented in this figure. The dashed orbit in the center of the figure represents that each orbit segment on the left can connect to each one on the right. For instance, in the case $q_0 \in \Sigma^{ss}$ and $q_1 \in \Sigma^{us}$, one must consider the red orbit connecting point q_0 in the top left of the figure to point q_1 in the bottom right. In this case, the bottom left and top right must be ignored

q_0' (that does not contain q_0) in such a way that every point of U_0 flows to Σ^{ss} and reaches q_0 (see Figs. 4 and 5).

If $q_0 \in \Sigma^{us}$, we can choose an open set U_0 in an analogous way, but now inverting the direction of the flow of time: we take a point q_0' after q_0 and an open set U_0 around it such that every point of U_0 must have come from Σ^{us} and, consequently, passed through q_0 (see Fig. 5). In the same way we choose an open set $U_1 \subset M$ for q_1.

By topological transitivity, there is a Filippov orbit segment from a point $p_0 \in U_0$ to a point of $p_1 \in U_1$. In the case that $q_0 \in \Sigma^{ss}$, by the choice of U_0 the orbit segment must have passed through q_0, so it can be restricted to an orbit segment from q_0 to p_1; in the case that $q_0 \in \Sigma^{us}$, by the choice of U_0 the point p_0 must have flowed away from q_0, so the orbit segment can also be extended to one from q_0 to p_1.

On the other end, the situation is inverted. If $q_1 \in \Sigma^{ss}$, the point p_1 must flow into Σ^{ss} and pass through q_1, so the orbit segment may be extended to one from q_0 to q_1; if $q_1 \in \Sigma^{ss}$, the point p_1 must have passed through q_1, so the orbit segment may also be extended to one from q_0 to q_1. □

The next lemma establishes some properties of the sets described in Definition 3.3.

Lemma 4.2 *Assume topological transitivity.*

1. *Suppose $\Sigma^{ss} \neq \emptyset$. Then E_+ is open and D_+ is dense.*
2. *Suppose $\Sigma^{us} \neq \emptyset$. Then E_- is open and D_- is dense.*

Proof We prove the first item, since the second is the same but with the direction of orbits inverted. Let $p \in E_+$, and let $q \in \Sigma^{ss}$ be the point where p first loses uniqueness going forward. Since Σ^{ss} is open in Σ and every point in the orbit from p to q is a regular point, we can find an open set around p that first loses uniqueness in Σ^{ss}. This shows E_+ is open.

Showing D_+ is dense is the same as showing $D_+^{\complement}$ has empty interior. For this, suppose for the sake of contradiction, that there is a non-empty open set $U \subset D_+^{\complement}$. Since $\Sigma^{ss} \neq \emptyset$, there is an open set V such that all of its points reach Σ^{ss} (see Fig. 4). By topological transitivity, there is an orbit from U to V, and so this orbit can be extended to reach Σ^{ss}, which contradicts the fact that U is a set of points that do not reach Σ^{ss} by any orbit. □

In the known examples of transitive Filippov systems in the literature, the bean model and the sphere model [1, 5, 13], it is easy to check that the sets E_+ and E_- are dense, so in the following results (Lemma 4.3, Theorem 2.2) we will assume this in order to obtain dense periodic orbits (for further discussion, check Subsection 4.1).

Lemma 4.3 *Assume topological transitivity, and that either $\Sigma^{ss} \neq \emptyset$ and E_+ is dense, or that $\Sigma^{us} \neq \emptyset$ and E_- is dense. Let $q_0 \in \Sigma^s$ and $U \subseteq M$ be a non-empty open set. Then there is a periodic Filippov orbit segment through q_0 that intersects U.*

Proof We consider the case in which $q_0 \in \Sigma^{ss}$. The proof for $\Sigma^{us} \neq \emptyset$ is the same after inverting the direction of the orbits. Since q_0 is in the sliding region, we may choose an open set $U_0 \subseteq M$ such that all of its points pass through q_0 going forwards, as done in Lemma 4.1 (see Fig. 6). Since the set E_+ is open and we assume it is dense, take $p \in E_+ \cap U$ and an open neighborhood $V \subseteq E_+$ of p.

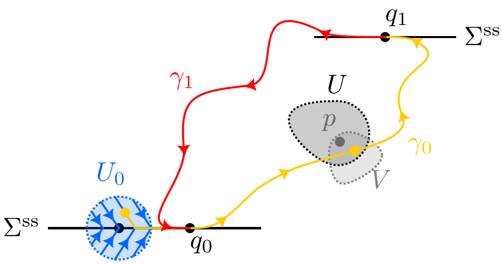

Fig. 6 Given a point $q_0 \in \Sigma^s$ and an open set U, we can find a periodic orbit segment starting at q_0 that intersects U. In the figure we depict the case in which $\Sigma^{ss} \neq \emptyset$

By topological transitivity, we choose a Filippov orbit segment γ_0 from U_0 to $U \cap V$. Since all points of U_0 pass through q_0, γ_0 can be restricted to start at q_0, and since γ_0 passes through $V \subseteq E_+$, it can be extended to reach a point $q_1 \in \Sigma^{ss}$. Now, from Lemma 4.1, there is an orbit segment γ_1 from q_1 to q_0. Concatenating the orbit segments γ_0 (restricted to start at q_0) and γ_1, we obtain a periodic orbit segment starting at q_0 that intersects U. □

Finally, we can prove Theorem 2.2.

Proof of Theorem 2.2 *We prove each item separately.*

1. *Take $x \in \Sigma^s$.*

1.1 *Let $\{U_i\}_{i\in\mathbb{N}}$ be a countable base of M. By Lemma 4.3, for each U_i there is a periodic Filippov orbit segment γ_i that starts at x and intersects U_i. We denote the concatenation of the orbit segments γ_i in the increasing order of indices by $\gamma_+ := \gamma_0 \cdot \gamma_1 \cdot \gamma_2 \cdots$ (this is the trajectory obtained by concatenating γ_0 to γ_1 to obtain $\gamma_0 \cdot \gamma_1$, then $\gamma_0 \cdot \gamma_1$ to γ_2, and so on for every $i \in \mathbb{N}$) and the concatenation of the orbit segments γ_i in the decreasing order of indices by $\gamma_- := \cdots \gamma_2 \cdot \gamma_1 \cdot \gamma_0$, and define γ to be the concatenation of these trajectories*:

$$\gamma := \gamma_- \cdot \gamma_+ = \cdots \gamma_2 \cdot \gamma_1 \cdot \gamma_0 \cdot \gamma_0 \cdot \gamma_1 \cdot \gamma_2 \cdots .$$

Now for every non-empty open set $U \subseteq M$, there is an open set U_i contained in U, so γ intersects U because γ_i intersects U_i. This shows γ is dense in M.

1.2 *Now let P be the union of (the image of) all the periodic orbits of Z through x. By Lemma 4.3 it follows that, for every non-empty open set $U \subseteq M$, there is a periodic orbit segment γ that passes through x and intersects U, hence a periodic orbit through x and U which means that $P \cap U \neq \emptyset$, so P is dense in M.*

We now take Δ as the union of all the points on periodic orbits as constructed in the preceding items.

2. *We will assume that $\Sigma^{ss} \neq \emptyset$, but the proof for $\Sigma^{us} \neq \emptyset$ is analogous. First, we take $q \in \Sigma^{ss}$ and two open sets U_0 and U_1, each one on each side of the connected component of Σ^{ss} on which q is (see Fig. 7), and not intersecting Σ^{ss}. Now we choose two different periodic orbits β_0 and β_1 starting at q, similarly to what was done in Lemma 4.3, but in this case we force each orbit to enter Σ^{ss} on the open sets U_0 and U_1, respectively, before they reach q. Denote their periods respectively by b_0 and b_1. Since Σ^{ss} is open, we may disturb these orbits slightly in order to have the ratio of their periods, $\frac{b_1}{b_0}$, be an irrational number. For each $i \in \{0, 1\}$, we choose a point $p_i \in U_0$ through which orbit β_i passes,*

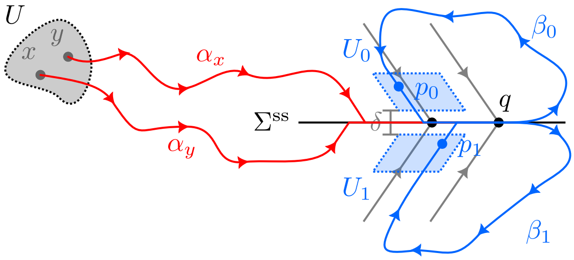

Fig. 7 For any non-empty open set U, there are points x and y and orbits γ_x and γ_y such that, following these orbits, the two points eventually become δ apart. The orbits are constructed by connecting the points to a sliding region and then following periodic orbits for some time

and define a restricted orbit β_i' from q to p_i, and denote its period b_i'. Denote $\delta := d(U_0, U_1)$, the infimum of the distance between any point of U_0 and any point of U_1. Since each open set has been taken on one side of Σ^{ss}, we have $\delta > 0$. Now let U be any non-empty open set. Since our set Δ as defined in the preceding item of this proof is dense, we may take $x, y \in \Delta \cap U$. Then there exist orbits $\alpha_x : [0, a_x] \to M$ and $\alpha_y : [0, a_y] \to M$ starting at x and y, respectively, and ending at q. Now we define orbit γ_x as the concatenation of α_x and $k_x \in \mathbb{N}$ orbits β_0, followed by the restricted orbit β_0'; that is, $\gamma_x := \alpha_x \cdot (\beta_0)^{k_x} \cdot \beta_0'$. Likewise, define $\gamma_y := \alpha_y \cdot (\beta_1)^{k_y} \cdot \beta_1'$, for $k_y \in \mathbb{N}$. We denote $c_x := a_x + k_x b_0 + b_0'$ and $c_y := a_y + k_y b_1 + b_1'$, which represent the time x (or y) takes to traverse γ_x (resp. γ_y) and reach p_0 (resp. p_1). Since the ratio $\frac{b_1}{b_0}$ is irrational, the integers k_x and k_y may be chosen such that c_x and c_y are as close as needed to guarantee that (supposing $c_x \leq c_y$ without loss of generality) $\gamma_x(c_x) = p_0$ and $\gamma_y(c_x) \in U_1$, close to p_1. Defining $t := c_x$, this implies that $d(\gamma_x(t), \gamma_y(t)) > \delta$.

□

Comments About Finding a Residual Set

In the classical case of smooth vector fields, transitivity is equivalent to the existence of a residual set of points through which there is a dense orbit. In our setting, we have found a dense set Δ instead. As mentioned after the proof of Lemma 4.2, for the bean model and the sphere model [5, 13] both sets E_+ and E_- are dense, hence open dense sets. So we pose the following question:

Question 2 *In the general context of a transitive Filippov system with a finite number of tangency points, are the sets E_+ and E_- dense?*

If both[2] E_+ and E_- were open dense sets, this would imply that their intersection $E := E_+ \cap E_-$ is a residual set (and hence a dense set by Baire category theorem). The proof of Lemma 4.3 can be easily adapted to be valid for points x in this residual set E (Fig. 8). This can be done by starting with a point $x \in E$ instead of $q_0 \in \Sigma^s$, and then connecting x to Σ^s forward and backwards, which is possible by definition of E. After that, the final details of the proof would be almost the same as in Lemma 4.3.

[2] Notice we are assuming here that both Σ^{ss} and Σ^{us} are non-empty.

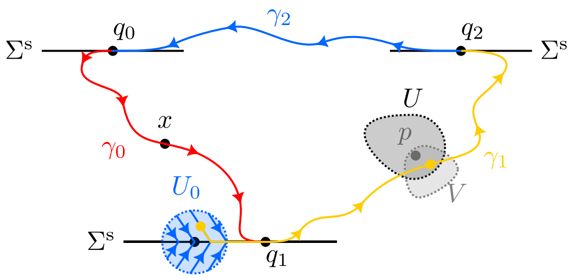

Fig. 8 Given a point $x \in \Delta$ and an open set U, we could find a periodic orbit segment starting at x that intersects U

Proof of Theorem 2.1

We now proceed to prove Theorem 2.1. If $\Sigma^s \neq \emptyset$, then the result follows from Theorem 2.2. The case $\Sigma^s = \emptyset$ will follow from the two lemmas below. We now assume that the Filippov system is topologically transitive, since the converse is trivial.

Recall that, as per Definition 3.2 of Sect. 3, the saturation of a set $A \subset M$, denoted A_φ, is the union of every Filippov orbit with initial condition on A.

Lemma 5.1 *Assume topological transitivity, $\Sigma^s = \emptyset$ and Σ^t_φ is dense. Then there exists a residual set of M such that each point in this set has some dense Filippov orbit.*

Proof Recall that we are assuming a finite number of tangency points and our space is two dimensional. Hence, we may construct a Filippov orbit inside Σ^t_φ which is dense.

Let us call this dense Filippov orbit $\gamma(t)$, and let t_0 be the time such that $\gamma(t_0) \in \Sigma^t$ and, for all $t > t_0$, $\gamma(t) \cap \Sigma^t = \emptyset$. Therefore $\gamma([t_0, \infty])$ is dense in M and it does not intersect any tangency point.

Let $g : M \to [0, 1]$ be a smooth function such that $g^{-1}(0) = \Sigma^t$. Now let us change the velocity of the Filippov orbit associated with the vector field of the system by multiplying it by the positive map g. Hence the new vector field gZ is in fact a continuous vector field whose orbits coincide with the previous one, with the exception that the tangency points are now fixed points.

But this new Filippov system is in fact a continuous flow, and $\gamma([t_0, \infty])$ is the image of some dense trajectory of this new flow. Thus we guarantee from classical results for continuous flows that there is a residual set D of points with dense orbit. Notice that none of these dense orbits pass through the singular points of gZ, hence they are also dense orbits for the Filippov system.

Finally, notice that Σ^t_φ is a meager set, because it is a countable union of orbit segments, therefore $D \setminus \Sigma^t_\varphi$ is a residual set and the image of the orbit of these points for the flow gZ is the same as for the original Filippov system. □

Lemma 5.2 *Assume topological transitivity, $\Sigma^s = \emptyset$ and Σ^t_φ is not dense. Then there is a residual set of M such that each point in this set has some dense orbit. Moreover, these dense orbits are regular orbits.*

Proof If $\Sigma^{\mathrm{t}}_{\varphi}$ is not dense, then the interior of its complement $\left(\Sigma^{\mathrm{t}}_{\varphi}\right)^{\complement}$ is non-empty. Let us take $\mathcal{U} \subset \operatorname{int}[\left(\Sigma^{\mathrm{t}}_{\varphi}\right)^{\complement}] =: A$. Now we prove that A is dense. Indeed, given an open set $\mathcal{V} \subset M$, from the topological transitivity of the Filippov system there is a point $x \in \mathcal{V}$ and an orbit from x to $\mathcal{U}$ which is regular since it is outside the set where the break of uniqueness occurs. Hence there is an open set $\mathcal{V}_0 \subset V$ containing x such that every segment of orbit from $\mathcal{V}_0$ to $\mathcal{U}$ is regular, which is true for $\mathcal{V}_0$ sufficiently small. Therefore from the invariance of the set A, we get that $\mathcal{V}_0 \subset A$. That means the Filippov system on A is topologically transitive, but on this invariant set A the Filippov system determines a continuous flow since $A \cap \Sigma \subset \Sigma^{\mathrm{c}}$ because $\Sigma^{\mathrm{s}} = \emptyset$. Thus there is a residual set of points whose orbit is dense in A.

The lemma is proved, since Z on A is a continuous flow, A is a dense open set, and Z restricted to A has a dense orbit. □

Proof of Theorem 2.1 *Since the case $\Sigma^{\mathrm{s}} \neq \emptyset$ is a consequence of Theorem 2.2, and Lemmas 5.1 and 5.2 encompass the case $\Sigma^{\mathrm{s}} = \emptyset$, we have proved the result.* □

Proof of Theorem 2.3

We will now estimate the entropy of our topologically transitive Filippov system Z to show it is strictly positive. We will use the orbit space $\widetilde{M}$ and its flow $\widetilde{\Phi}^t$, which is induced by Z. By Definition 3.7, the entropy of Z is the entropy of the time 1 map $\widetilde{\Phi}^1 : \widetilde{M} \to \widetilde{M}$. We will define a subset $\widetilde{\Gamma}_p$ of $\widetilde{M}$ and consider on it the induced flow, and show that its entropy is strictly positive, which implies that the entropy of $\widetilde{M}$ is also positive since the entropy of a subsystem is always smaller than the entropy of the system [23].

To define the subset $\widetilde{\Gamma}_p$, we first need to have two different periodic orbits of the Filippov system Z which have the same period and have a point in common. We start with a simple lemma that proves this is the case for our setting.

Lemma 6.1 *Assume topological transitivity and $\Sigma^{\mathrm{s}} \neq \emptyset$. There exist 2 distinct periodic Filippov orbit segments $\gamma_0, \gamma_1 : [0, \alpha] \to M$ of Z with period $\alpha > 0$ and initial point $p = \gamma_0(0) = \gamma_0(\alpha) = \gamma_1(0) = \gamma_1(\alpha) \in M$.*

Proof We can find two distinct periodic Filippov orbit segments η_0 and η_1 that have an initial point $p \in \Sigma^{\mathrm{ss}}$, in the same way it was done in Lemmas 4.1 and 4.3, and they can be forced to differ by making each one pass through each different side of the connected component of Σ^{s} being considered (see Fig. 4). Since the orbit segments may have different periods, to obtain γ_0 and γ_1 with the same period we concatenate η_0 and η_1 in the two possible orders. The resulting orbit segments have the same period (the sum of the periods of η_0 and η_1). □

Construction of the Subsystem

Now we present the construction of the subsystem $\widetilde{\Gamma}_p$ of the orbit space $\widetilde{M}$. We take the periodic orbit segments γ_0 and γ_1 as in Lemma 6.1. Let $\Gamma_0 := \gamma_0([0,\alpha])$ and $\Gamma_1 := \gamma_1([0,\alpha])$ denote their images, two curves on the base space M that are distinct and have their initial point p in common.[3] Denote the union of these curves as $\Gamma := \Gamma_0 \cup \Gamma_1$. The set $\widetilde{\Gamma}_p$ is the set of all the possible integral trajectories of Z that start at p and whose image lies on Γ.

All trajectories of $\widetilde{\Gamma}_p$ must start at p and follow either Γ_0 or Γ_1, returning to p after the period α has passed. Then, it must again follows either of the two curves and so on forwards and backwards in time. We can more formally describe this as follows. We define the set of indices $S := \{0, 1\}$ and let $S^{\mathbb{Z}}$ denote the space of all sequences $x : \mathbb{Z} \to S$, that is, sequences

$$x = (\dots, x_{-1}; x_0, x_1, \dots)$$

such that $x_i \in S$. The shift map on $S^{\mathbb{Z}}$ is defined as the map $\sigma : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ such $\sigma(x) = (\dots, x_0; x_1, x_2, \dots)$; i.e., $\sigma(x)_i = x_{i+1}$ for each $i \in \mathbb{Z}$. For each sequence $x \in S^{\mathbb{Z}}$, we can define an orbit γ_x of Z (which starts at p and whose image lies on Γ) as the concatenation of the orbit segments γ_0 and γ_1 according to the entries of x: for each $i \in \mathbb{Z}$ and each $t \in [i\alpha, (i+1)\alpha[$, we have $\gamma_x(t) = \gamma_{x_i}(t - i\alpha)$. We will denote this by[4]

$$\gamma_x(t) = \left\{ \gamma_{x_i}(t - i\alpha),\ i\alpha \le t < (i+1)\alpha. \right.$$

This gives the characterization $\widetilde{\Gamma}_p = \left\{ \gamma_x \mid x \in S^{\mathbb{Z}} \right\}$.

To define the dynamics on $\widetilde{\Gamma}_p$, we take the map $\widetilde{\Phi}^\alpha : \widetilde{\Gamma}_p \to \widetilde{\Gamma}_p$ given on each $\gamma \in \widetilde{\Gamma}_p$ by $\widetilde{\Phi}^\alpha(\gamma)(t) = \gamma(\alpha + t)$. This is the flow map $\widetilde{\Phi}^\alpha : \widetilde{M} \to \widetilde{M}$ (as defined in Subsection 3.3) restricted to $\widetilde{\Gamma}_p$. The restriction is well defined because $\widetilde{\Phi}^\alpha(\gamma_x) = \gamma_{\sigma(x)} \in \widetilde{\Gamma}_p$ for every $x \in S^{\mathbb{Z}}$; this follows from the calculation

$$\begin{aligned}
\widetilde{\Phi}^\alpha(\gamma_x)(t) &= \gamma_x(t + \alpha) \\
&= \left\{ \gamma_{x_i}(t + \alpha - i\alpha),\ i\alpha \le t + \alpha < (i+1)\alpha \right. \\
&= \left\{ \gamma_{x_i}(t - (i-1)\alpha),\ (i-1)\alpha \le t < i\alpha \right. \\
&= \left\{ \gamma_{x_{i+1}}(t - i\alpha),\ i\alpha \le t < (i+1)\alpha \right. \\
&= \gamma_{\sigma(x)}(t).
\end{aligned}$$

This shows that $\widetilde{\Phi}^\alpha : \widetilde{\Gamma}_p \to \widetilde{\Gamma}_p$ is a subsystem of $\widetilde{\Phi}^\alpha : \widetilde{M} \to \widetilde{M}$.

Entropy Calculation

We are ready to calculate the entropy. Let us define the constant

[3] Notice that in the following constructions and propositions, we only need the existence of 2 distinct periodic orbits γ_0 and γ_1 that have the same initial point p and same period α. Even though in the proof of Lemma 6.1 such curves have the same image, in the general case of Lemma 6.2 and Proposition 6.3 they do not need to.

[4] The use of curly braces is motivated by the similar notation used in the definition of functions by cases, for instance Eq. 1.

$$\mu := \sup_{0 \le t < \alpha} d(\gamma_0(t), \gamma_1(t)),$$

which is the maximum distance points of γ_0 and γ_1 may be from each other at the same time. Since γ_0 and γ_1 are not equal for all times, $\mu > 0$. This will be used in the estimates that follow.

Also, we define the quantity

$$N(\gamma_x, \gamma_{x'}) := N(x, x') := \min \left\{ |i| \;\middle|\; x_i \neq x'_i \right\}.$$

between two orbits γ_x and $\gamma_{x'}$. The quantity $N(x, x')$ is related to the distance function for the symbolic space $S^{\mathbb{Z}}$, and $N(\gamma_x, \gamma_{x'})$ serves a similar purpose in helping to estimate orbital distances, as the next lemma shows.

Lemma 6.2 *Let $m \in \mathbb{N}$ and $x, x' \in S^{\mathbb{Z}}$. If $N(\gamma_x, \gamma_{x'}) \le m$, then*

$$d_{\sup}(\gamma_x, \gamma_{x'}) \ge \mu 2^{-(m+1)\alpha}.$$

Proof Denote $N := N(\gamma_x, \gamma_{x'})$. By the definition of N, the orbits γ_x and $\gamma_{x'}$ are different on the interval $[N\alpha, (N+1)\alpha[$ or on the interval $[-N\alpha, -(N-1)\alpha[$. Denote $u_i := \sup_{i \le t < i+1} d(\gamma_x(t), \gamma_{x'}(t))$. Then $u_j = \mu$ for some $j \in \mathbb{N}$ such that $\lfloor N\alpha \rfloor \le j < \lfloor (N+1)\alpha \rfloor$, so

$$d_{\sup}(\gamma_x, \gamma_{x'}) = \sum_{i \in \mathbb{Z}} 2^{-|i|} u_i \ge \sum_{i=\lfloor N\alpha \rfloor}^{\lfloor (N+1)\alpha \rfloor} 2^{-i} u_i \ge 2^{-j}\mu \ge 2^{-(m+1)\alpha}\mu.$$

□

Finally, we calculate the entropy of the subsystem $\widetilde{\Phi}^\alpha : \widetilde{\Gamma}_p \to \widetilde{\Gamma}_p$.

Proposition 6.3 $h(\widetilde{\Phi}^\alpha|_{\widetilde{\Gamma}_p}) \ge \log(2)$.

Proof To simplify notation, we will just write $\widetilde{\Phi}^\alpha$ instead of $\widetilde{\Phi}^\alpha|_{\widetilde{\Gamma}_p}$ inside this proof. (Check Subsection 3.4 for definitions and notation.) We will first show that $g^n(\widetilde{\Phi}^\alpha, \mu 2^{-(m+1)\alpha}) \ge 2^{2m+n}$. Let $E \subseteq \widetilde{\Gamma}_p$ be a set of orbits γ_y such that $\#E \le 2^{2m+n} - 1$. Consider the set of $(2m+n)$-tuples

$$F := \left\{ (y_{-m}, \dots, y_{m+n-1}) \in S^{2m+n} \;\middle|\; \gamma_y \in E \right\}.$$

Since there are at most $2^{2m+n} - 1$ elements in E, then $\#F \le 2^{2m+n} - 1$; and since $\#(S^{2m+n}) = 2^{2m+n}$, there is at least one $(2m+n)$-tuple $(s_{-m}, \dots, s_{m+n-1}) \in S^{2m+n} \setminus F$. Take a sequence $x \in S^{\mathbb{Z}}$ such that

$$(x_{-m}, \dots, x_{m+n-1}) = (s_{-m}, \dots, s_{m+n-1}).$$

By the choice of the s_i, it follows that, for every $\gamma_y \in E$, $(y_{-m}, \dots, y_{m+n-1}) \neq (x_{-m}, \dots, x_{m+n-1})$. So there exist $l_y \in \{-m, \dots, m\}$ and $k_y \in \{0, \dots, n-1\}$ such that $y_{l_y + k_y} \neq x_{l_y + k_y}$, which is the same as $\sigma^{k_y}(y)_{l_y} \neq \sigma^{k_y}(x)_{l_y}$. Since $\left| l_y \right| \le m$, this implies that $N(\widetilde{\Phi}^{\alpha k_y}(\gamma_y), \widetilde{\Phi}^{\alpha k_y}(\gamma_x)) = N(\sigma^{k_y}(y), \sigma^{k_y}(x)) \le m$, so it follows from Lemma 6.2 that

$$d_{\sup^n_{\widetilde{\Phi}^\alpha}}(\gamma_y, \gamma_x) \geq d_{\sup}(\widetilde{\Phi}^{\alpha k_y}(\gamma_y), \widetilde{\Phi}^{\alpha k_y}(\gamma_x)) \geq \mu 2^{-(m+1)\alpha};$$

that is, $\gamma_x \notin B^n_{\widetilde{\Phi}^\alpha}(\gamma_y, \mu 2^{-(m+1)\alpha})$. Since this is valid for every $\gamma_y \in E$, we conclude that $\gamma_x \notin \bigcup_{\gamma_y \in E} B^n_{\widetilde{\Phi}^\alpha}(\gamma_y, \mu 2^{-(m+1)\alpha})$ and, because E is an arbitrary set with $\#E \leq 2^{2m+n} - 1$, it follows that $g^n(\widetilde{\Phi}^\alpha, \mu 2^{-(m+1)\alpha}) > 2^{2m+n} - 1$.

Finally, to estimate the entropy we note that, since $\mu 2^{-(m+1)\alpha} \to 0$ as $m \to \infty$, and

$$\lim_{m\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log 2^{2m+n} = \lim_{m\to\infty} \limsup_{n\to\infty} \frac{2m+n}{n} \log(2) = \log(2),$$

so it follows from $g^n(\widetilde{\Phi}^\alpha, \mu 2^{-(m+1)\alpha}) \geq 2^{2m+n}$ that $h(\widetilde{\Phi}^\alpha) \geq \log(2)$. □

We can now finish the proof of the main entropy theorem.

Proof of Theorem 2.3 *From Lemma 6.1 we know that our Filippov system Z has two periodic orbit segments γ_0 and γ_1, so the construction of the $\widetilde{\Gamma}_p$ can be done. From Proposition 6.3, we have that $h(\widetilde{\Phi}^\alpha|_{\widetilde{\Gamma}_p}) \geq \log(2)$. Since $\widetilde{\Phi}^\alpha : \widetilde{\Gamma}_p \to \widetilde{\Gamma}_p$ is a subsystem of $\widetilde{\Phi}^\alpha : \widetilde{M} \to \widetilde{M}$, this means that $h(\widetilde{\Phi}^\alpha) \geq h(\widetilde{\Phi}^\alpha|_{\widetilde{\Gamma}_p})$. From the fact that $h(\widetilde{\Phi}^\alpha) = \alpha h(\widetilde{\Phi}^1)$ (the exponent property of entropy), it follows that*

$$h(Z) = h(\widetilde{\Phi}^1) = \alpha^{-1} h(\widetilde{\Phi}^\alpha) \geq \alpha^{-1} \log(2) > 0.$$

□

Numerical examples for the calculation of topological entropy of Filippov system can be found in [1], in which the bean model and other examples are analyzed.

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