Piecewise Smooth Vector Fields

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Contents

1 Preliminaries

The *empty set* is denoted $\mathbb{0}$, the *point set*, $\mathbb{1} = \{0\}$, and $\mathbb{2} = \{0, 1\}$. Given a 'universe' set X, the *power set* of X (the set of all its subsets) is denoted 2^X, and the *set complement* (*in* X) of a subset $S \subseteq X$ is denoted $\overline{S} = X \setminus S$. The identity function in X is $\mathrm{I} = \mathrm{I}_X : X \to X$.

The *natural* numbers are denoted $\mathbb{N}$ and include 0 (zero), and, for every natural number $n \in \mathbb{N}$, the set of the first n natural numbers is denoted $[\![n]\!] := \{n' \in \mathbb{N} \mid n' < n\} = \{0, \dots, n-1\}$. The *integers* are denoted $\mathbb{Z}$, the *rational* numbers, $\mathbb{Q}$, and the *real* numbers, $\mathbb{R}$. The *positive*, *strictly positive*, *negative* and *strictly negative* integers are denoted $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\leq 0}$ and $\mathbb{Z}_{<0}$, respectively. Analogous notation is used for other number sets. We denote the *infimum* of a set S by $\inf S$ and its supremum by $\sup S$.

We denote a *topological space* by $\boldsymbol{X} = (X, \mathcal{T})$, being X the set of points of the space and $\mathcal{T}$ its topology[1]. We denote the *topological interior* of a set $S \subseteq X$ by S° and its *topological closure* by $S^\bullet$.

1.1 Manifolds, vector fields and flows

We will restrict our analysis to finite dimensional manifolds. We denote a *d-dimensional $\mathscr{C}^m$-smooth manifold* by $\boldsymbol{M} = (M, \mathcal{A})$, being M the set of points of the manifold and $\mathcal{A}$ its d-dimensional $\mathscr{C}^m$-smooth atlas. The *boundary* of $\boldsymbol{M}$ is a $(d-1)$-dimensional $\mathscr{C}^m$-smooth submanifold of $\boldsymbol{M}$ denoted $\partial\boldsymbol{M}$.

The *tangent space* of $\boldsymbol{M}$ at a point $p \in M$ is a d-dimensional vector space denoted $\mathrm{T}M|_p$, and the *tangent bundle* of $\boldsymbol{M}$ is a $2d$-dimensional $\mathscr{C}^m$-smooth vector bundle over $\boldsymbol{M}$ denoted $\mathrm{T}\boldsymbol{M}$. We denote the *differential* of a differentiable transformation $f: M \to M'$ at a point $p \in M$ by $\mathrm{D}f|_p : \mathrm{T}M|_p \to \mathrm{T}M'|_{f(p)}$; the *derivative* of a differentiable trajectory $\gamma: I \to M$ at a time $t_0 \in I$ by

$$\dot{\gamma}(t_0) = \dot{\partial}\gamma(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)|_{t=t_0},$$

and the *exterior derivative* of a *k-form field* ω by $\mathrm{d}\omega$.

Given a $\mathscr{C}^m$-smooth vector field $v: M \to \mathrm{T}M$, its *flow* at a point $p \in M$ and time $t \in \mathbb{R}$ is denoted $\Phi^{tv}(p)$ and its time derivative, $\dot{\Phi}^{tv}(p) := \frac{\mathrm{d}}{\mathrm{d}s}\Phi^{sv}(p)|_{s=t}$. When the vector field is complete and its flow is global, we have a function

$$\begin{aligned} \Phi^v : \mathbb{R} \times M &\longrightarrow M \\ (t, p) &\longmapsto \Phi^{tv}(p) \end{aligned}$$

and, for every $t \in \mathbb{R}$, a $\mathscr{C}^{m+1}$-diffeomorphism

$$\begin{aligned} \Phi^{tv} : M &\longrightarrow M \\ p &\longmapsto \Phi^{tv}(p). \end{aligned}$$

This notation suggestes many properties the flow Φ^{tv} shares with the exponential function. By the definition of the flow, we have that

$$\begin{aligned} \Phi^{0v} &= \mathrm{I}, \\ \dot{\Phi}^{tv} &= v \circ \Phi^{tv}. \end{aligned}$$

Besides that, by the scalling lemma, the flow of the vector field v at time t equals the flow of the vector field tv at time 1:

$$\Phi^{tv} = \Phi^{1(tv)},$$

and, by the translation lemma, the flow of v at time $t + t'$ is the composition of the flow of v at times t and t':

$$\Phi^{(t+t')v} = \Phi^{tv} \circ \Phi^{t'v}.$$

Also, if $\mathscr{C}^m$-smooth vector fields v and v' commute ($[v, v'] = 0$), then

$$\Phi^{t(v+v')} = \Phi^{tv} \circ \Phi^{tv'},$$

but it is *important* to notice, though, that this last formula is *not* always true for vector fields that do not commute.

1. We generally use boldface to indicate the space/categorial object $\boldsymbol{X}$, including its structure (its topology $\mathcal{T}$, distance $|\cdot,\cdot|$, binary operation $+$, ...), while the normal weight font denotes the underlying set X.

The (Lie) flow derivative of a $\mathscr{C}^{m+1}$-smooth scalar field $f\colon M \to \mathbb{R}$ by the flow of a $\mathscr{C}^{m+1}$-smooth vector field $v\colon M \to \mathrm{T}M$ at a point $p \in M$ is

$$\partial_v f(p) := \lim_{t\to 0} \frac{f(\Phi^{tv}(p)) - f(p)}{t}.$$

The derivative of f by the flow of v is a $\mathscr{C}^m$-smooth scalar field $\partial_v f\colon M \to \mathbb{R}$, which can be further differentiated along the flow of a vector field. If we differentiate f by the same vector field v a number $n \in \mathbb{N}$ of times, we denote it by $\partial_v^n f$ (taking $\partial_v^0 = \mathrm{I}$, the identity). The flow derivative can be related to the exterior as follows.

$$\begin{aligned}
\partial_v f(p) &= \lim_{t\to 0} \frac{f(\Phi^{tv}(p)) - f(p)}{t}\\
&= \lim_{t\to 0} \frac{f(\Phi^{tv}(p)) - f(\Phi^{0v}(p))}{t}\\
&= \frac{\mathrm{d}}{\mathrm{d}t} f \circ \Phi^{tv}(p)|_{t=0}\\
&= \mathrm{D}f|_{\Phi^{tv}(p)}(\dot{\Phi}^{tv}(p))|_{t=0}\\
&= \mathrm{D}f|_p(v(p))\\
&= \mathrm{d}f|_p(v(p)).
\end{aligned}$$

When the manifold $\boldsymbol{M}$ is given a (Riemannian) metric field $\langle\,\cdot\,,\cdot\,\rangle$, the *gradient* ∇ of a scalar field f is defined by duality with the exterior derivative d as $\langle \nabla f(p)|_p := \mathrm{d}f|_p$, which means that, for every tangent vector $v \in \mathrm{T}M|_p$,

$$\mathrm{d}f|_p(v) = \langle \nabla f(p), v\rangle_p.$$

In this case, the flow derivative $\partial_v f(p)$ can be expressed using the gradient of the scalar field f as

$$\partial_v f(p) = \langle \nabla f(p), v(p)\rangle_p.$$

2 Piecewise smooth vector fields

Definition 2.1. Let $\boldsymbol{M}$ be a d-dimensional $\mathscr{C}^{m+1}$-manifold. A *switching submanifold* of $\boldsymbol{M}$ is an embedded 1-codimensional $\mathscr{C}^{m+1}$-submanifold Σ of $\boldsymbol{M}$ for which there exists an open neighborhood $U_\Sigma \subseteq M$ of Σ in M and a $\mathscr{C}^{m+1}$-smooth function $h\colon U_\Sigma \to \mathbb{R}$ such that 0 is a regular value of h and $h^{-1}(0) = \Sigma$.

1. The *upper neighborhood* (or *upper side*) of Σ is $\Sigma_+ := h^{-1}(\mathbb{R}_{>0})$
2. The *lower neighborhood* (or *lower side*) of Σ is $\Sigma_- := h_i^{-1}(\mathbb{R}_{<0})$.

The *regular region* of M relative to Σ is the open $\mathscr{C}^{m+1}$-submanifold $M \smallsetminus \Sigma$. A *switching* point is a point in $p \in \Sigma$ and a *regular* point is a point in $M \smallsetminus \Sigma$.

Question 2.1. Can an embedded 1-codimensional connected submanifold of $\boldsymbol{M}$ always be given as the preimage of a regular value of a $\mathscr{C}^{m+1}$-smooth function? Is this condition necessary in the definition of a switching manifold?

This means that a piecewise smooth vector field $v\colon M \to \mathrm{T}M$ is a $\mathscr{C}^{m+1}$-smooth vector field on the regular region $M \smallsetminus \Sigma$, and may have discontinuities only on the switching manifold Σ. We will never take into consideration the value of v on Σ, but rather define a diferent vector field on part of it, called the sliding vector field, and ignore its definition on the rest of it.

2.1 Regions of the switching manifold

The (Lie) flow derivative of a function $h\colon M \to \mathbb{R}$ by the flow of a vector field $v\colon M \to \mathrm{T}M$ at a point $p \in M$ is

$$\partial_v h(p) := \lim_{t\to 0} \frac{h(\Phi^{tv}(p)) - h(p)}{t}.$$

The flow derivative of h by the flow of v is another function $\partial_v h\colon M \to \mathbb{R}$, which can be further differentiated along the flow of a vector field. If we differentiate by the same vector field v a number n of times, we denote $\partial_v^n h$.

When the manifold $\boldsymbol{M}$ is given a (Riemannian) metric field $\langle \cdot, \cdot \rangle$, the flow derivative $\partial_v h(p)$ can be expressed using the gradient of the function h as

$$\partial_v h(p) = \langle \nabla h(p), v(p) \rangle_p.$$

2.1.1 Order of contact with the switching manifold

We wish to classify the points of the switching manifold Σ acording to how the trajectories of the flows of the fields upper and lower vector fields v_+ and v_- intersect Σ. To this end, let $p \in U_\Sigma$ be a point in the neighborhood of Σ and consider the the real function

$$h_+(t) := h(\Phi^{tv_+}(p)),$$

which is defined for t in an ε-neighborhood $]-\varepsilon, \varepsilon[$ of 0.

This function gives the 'height' of the trajectory of p by the flow of v_+ from the switching manifold Σ, in which the height is interpreted according to the function h. Its derivative gives us information of how this trajectory intersects Σ at the point p. It can be expressed relatively to the derivative of h by the flow of v_+ as follows.

The derivative of h_+ is

$$\dot{h}_+(t) = \mathrm{D}h|_{\Phi^{tv_+}(p)} \left(\frac{\mathrm{d}}{\mathrm{d}t} \Phi^{tv_+}(p) \right) = \partial_{v_+} h(\Phi^{tv_+}(p)),$$

so its derivative at $t = 0$ is $\dot{h}_+(0) = \partial_{v_+} h(p)$. By induction we obtain that its n-th derivative at $t = 0$ is

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} \right)^n h_+(0) = \partial_{v_+}^n h(p).$$

The same can be done for the lower vector field v_- using the function

$$h_-(t) := h(\Phi^{tv_-}(p)).$$

We are ready to define the order of contact of the point p.

Definition 2.2. Let $\boldsymbol{M}$ be a $\mathscr{C}^{m+1}$-manifold, Σ a switching submanifold of $\boldsymbol{M}$, v a Σ-piecewise $\mathscr{C}^{m+1}$-smooth vector field on $\boldsymbol{M}$ and $p \in U_\Sigma$ a point in the neighborhood of Σ. The *order of upper Σ-contact* (or *upper multiplicity*) of p is

$$c_+ := \inf\{c \in \mathbb{N} \mid \partial_{v_+}^c h(p) \neq 0\}$$

and the *order of lower Σ-contact* (or *lower multiplicity*) of p is

$$c_- := \inf\{c \in \mathbb{N} \mid \partial_{v_-}^c h(p) \neq 0\}.$$

Since $\partial^0_{v_+} h(p) = \partial^0_{v_-} h(p) = h(p)$, points which are in the regular region $M \setminus \Sigma$ (more specifically in $\Sigma_+ \cup \Sigma_-$, the upper and lower regions of Σ) are the points that have both orders of contact equal to 0, and points which belong to Σ have both orders of contact greater than 0. A singularity p of the upper vector field v_+ satisfies $v_+(p) = 0$, so its flow trajectory is stationary. This implies that, for every $m \in \mathbb{N}_{>0}$, $\partial^m_{v_i} h(p) = 0$. If $p \in \Sigma$, this means that its order of upper contact is ∞. The same goes for the lower vector field v_-.

Question 2.2. Is it possible to have a point $p \in \Sigma$ that has order of upper/lower contact equal to ∞ but is not a singularity of the upper/lower vector field?

On the next subsections, we classify the points with finite positive order of contact according to their contact with the switching manifold from each of the positive and negative sides.

2.1.2 Crossing region

Definition 2.3 (Crossing region). Let $\boldsymbol{M}$ be a $\mathscr{C}^{m+1}$-manifold, Σ a switching submanifold of $\boldsymbol{M}$ and v a Σ-piecewise $\mathscr{C}^{m+1}$-smooth vector field on $\boldsymbol{M}$. The *crossing region* of Σ is the set

$$\Sigma^{\mathrm{c}} := \{p \in \Sigma \mid \partial_{v_+} h(p) \partial_{v_-} h(p) > 0\}.$$

A *crossing point* is a point $p \in \Sigma^{\mathrm{c}}$. The crossing region Σ^{c} can be divided into two regions.

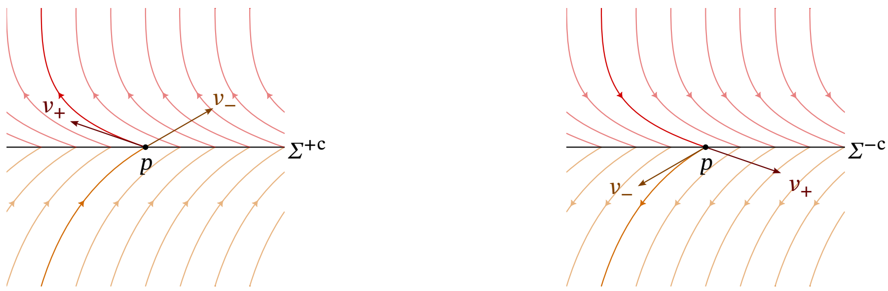
1. The *upward crossing region* of Σ is the set

$$\Sigma^{+\mathrm{c}} := \{p \in \Sigma^{\mathrm{c}} \mid \partial_{v_+} h(p) > 0\} = \{p \in \Sigma^{\mathrm{c}} \mid \partial_{v_-} h(p) > 0\}.$$

   An *upward crossing point* is a point $p \in \Sigma^{+\mathrm{c}}$.
2. The *downward crossing region* of Σ is the set

$$\Sigma^{-\mathrm{c}} := \{p \in \Sigma^{\mathrm{c}} \mid \partial_{v_-} h(p) < 0\} = \{p \in \Sigma^{\mathrm{c}} \mid \partial_{v_+} h(p) < 0\}.$$

   A *downward crossing point* is a point $p \in \Sigma^{-\mathrm{c}}$.



(1) An upward crossing point $p \in \Sigma^{+\mathrm{c}}$. **(2)** A downward crossing point $p \in \Sigma^{-\mathrm{c}}$.

Figure 1. The crossing region Σ^{c}.

2.1.3 Sliding region

Definition 2.4 (Sliding region). Let $\boldsymbol{M}$ be a $\mathscr{C}^{m+1}$-manifold, Σ a switching submanifold of $\boldsymbol{M}$ and v a Σ-piecewise $\mathscr{C}^{m+1}$-smooth vector field on $\boldsymbol{M}$. The *sliding region* of Σ is the set

$$\Sigma^{\mathrm{s}} := \{p \in \Sigma \mid \partial_{v_+} h(p) \partial_{v_-} h(p) < 0\}.$$

A *sliding point* is a point $p \in \Sigma^{\mathrm{s}}$. The sliding region Σ^{s} can be divided into two regions.

1. The *unstable sliding region* (or *escaping region*) of Σ is the set

$$\Sigma^{\mathrm{us}} := \{p \in \Sigma^{\mathrm{s}} \mid \partial_{v_+} h(p) > 0\} = \{p \in \Sigma^{\mathrm{s}} \mid \partial_{v_-} h(p) < 0\}.$$

   An *unstable sliding point* (or *escaping point*) is a point in $p \in \Sigma^{\mathrm{us}}$.
2. The *stable sliding region* (or *accessing region*) of Σ is the set

$$\Sigma^{\mathrm{ss}} := \{p \in \Sigma^{\mathrm{s}} \mid \partial_{v_-} h(p) > 0\} = \{p \in \Sigma^{\mathrm{s}} \mid \partial_{v_+} h(p) < 0\}.$$

   A *stable sliding point* (or *accessing point*) is a point in $p \in \Sigma^{\mathrm{ss}}$.

(1) An unstable sliding point $p \in \Sigma^{\mathrm{us}}$. **(2)** A stable sliding point $p \in \Sigma^{\mathrm{ss}}$.

Figure 2. The sliding region Σ^{s}.

2.1.4 Tangency region

Definition 2.5 (Tangency region)**.** Let $\boldsymbol{M}$ be a $\mathscr{C}^{m+1}$-manifold, Σ a switching submanifold of $\boldsymbol{M}$ and v a Σ-piecewise $\mathscr{C}^{m+1}$-smooth vector field on $\boldsymbol{M}$. The *tangency region* of Σ is the set

$$\Sigma^{\mathrm{t}} := \{p \in \Sigma \mid \partial_{v_+} h(p) \partial_{v_-} h(p) = 0\}.$$

A *tangency point* is a point $p \in \Sigma^{\mathrm{t}}$. The tangency region Σ^{t} can be divided into two regions.

1. The *simple tangency region* of Σ is the set

$$\Sigma^{\mathrm{st}} := \{p \in \Sigma^{\mathrm{t}} \mid \partial_{v_+} h(p) \neq 0\} \cup \{p \in \Sigma^{\mathrm{t}} \mid \partial_{v_-} h(p) \neq 0\}.$$

   A *simple tangency point* is a point $p \in \Sigma^{\mathrm{st}}$.
2. The *double tangency region* of Σ^{t} is the set

$$\Sigma^{\mathrm{dt}} := \{p \in \Sigma^{\mathrm{t}} \mid \partial_{v_+} h(p) = 0 = \partial_{v_-} h(p)\}.$$

   A *double tangency point* is a point $p \in \Sigma^{\mathrm{dt}}$.

Folds and cusps, visible and invisible We now classificate tangency points according to whether they can be reached with the flow of the vector field on the regular region.

Definition 2.6 (Visible and invisible tangencies)**.** Let $\boldsymbol{M}$ be a $\mathscr{C}^{m+1}$-manifold, Σ a switching submanifold of $\boldsymbol{M}$ and v a Σ-piecewise $\mathscr{C}^{m+1}$-smooth vector field on $\boldsymbol{M}$.

1. An (*upper*/*lower*) *visible* tangency point is a tangency point $p \in \Sigma^{\mathrm{t}}$ for which there exists $\varepsilon \in \mathbb{R}_{>0}$ such that at least one of the following holds:

1.1. (*Accessible*) For every $t \in \left]-\varepsilon, 0\right[$, $\Phi^{t v_\pm}(p) \in \Sigma_\pm$.
1.2. (*Escapable*) For every $t \in \left]0, \varepsilon\right[$, $\Phi^{t v_\pm}(p) \in \Sigma_\pm$.

A *visible* tangency point is tangency point that is upper visible or lower visible.

2. An (*upper*/*lower*) *invisible* tangency point is a tangency point $p \in \Sigma^{\mathrm{t}}$ for which there exists $\varepsilon \in \mathbb{R}_{>0}$ such that, for every $t \in \left]-\varepsilon, \varepsilon\right[\setminus \{0\}$, $\Phi^{t v_\pm}(p) \in \Sigma_\mp$. An *invisible* tangency point is a tangency point that is both upper invisible and lower invisible.

Definition 2.7 (Fold and cusp tangencies)**.** Let $\boldsymbol{M}$ be a $\mathscr{C}^{m+1}$-manifold, Σ a switching submanifold of $\boldsymbol{M}$ and v a Σ-piecewise $\mathscr{C}^{m+1}$-smooth vector field on $\boldsymbol{M}$.

1. An (*upper*/*lower*) *fold* tangency point is a tangency point $p \in \Sigma^{\mathrm{t}}$ whose order of (upper/lower) contact $c_\pm$ with Σ is even.
2. An (*upper*/*lower*) *cusp* tangency point is a tangency point $p \in \Sigma^{\mathrm{t}}$ whose order of (upper/lower) contact $c_\pm$ with Σ is odd.

The function $\partial_{v_\pm} h$ has the same sign at each side of a cusp tangency point, and has opposite signs at each side of a fold tangency point. A simple analysis of the concavity of the functions $h_\pm(t) = h(\Phi^{t v_\pm}(p))$ at 0 shows the following properties of folds and cusps.

Proposition 2.1. *Let $\boldsymbol{M}$ be a $\mathscr{C}^{m+1}$-manifold, Σ a switching submanifold of $\boldsymbol{M}$ and v a Σ-piecewise $\mathscr{C}^{m+1}$-smooth vector field on $\boldsymbol{M}$.*

1. *An (upper/lower) fold tangency point $p \in \Sigma^{\mathrm{t}}$ with order of upper contact $c_\pm$ is (upper/lower) visible (both accessible and escapable) if $\pm\partial_{v_\pm}^{c_\pm} h(p) > 0$, and upper invisible if $\pm\partial_{v_\pm}^{c_\pm} h(p) < 0$.*
2. *An (upper/lower) cusp tangency point is (upper/lower) visible. It is accessible if $\pm\partial_{v_\pm} h(p) < 0$, and escapable if $\pm\partial_{v_\pm} h(p) > 0$.*

	Visible		Invisible
	Accessible	Escapable	
Fold	$\pm\partial_{v_\pm}^{c_\pm} h(p) > 0$		$\pm\partial_{v_\pm}^{c_\pm} h(p) < 0$
Cusp	$\pm\partial_{v_\pm}^{c_\pm} h(p) < 0$	$\pm\partial_{v_\pm}^{c_\pm} h(p) > 0$	—

Table 1. Visibility of fold and cusp tangency points given by PROPOSITION 2.1. The sign $\pm$ indicates the side of Σ relative to which the tangency point p is a visible/invisible fold/cusp.

Definition 2.8 (Accessible and escapable points)**.** Let $\boldsymbol{M}$ be a $\mathscr{C}^{m+1}$-manifold, Σ a switching submanifold of $\boldsymbol{M}$ and v a Σ-piecewise $\mathscr{C}^{m+1}$-smooth vector field on $\boldsymbol{M}$.

1. An *upper* (resp. *lower*/*centrally*) *accessible* switching point is a switching point $p \in \Sigma$ for which there exists $\varepsilon \in \mathbb{R}_{>0}$ such that, for every $t \in \left]-\varepsilon, 0\right[$, $\Phi^{t v_+}(p) \in \Sigma_+$ (resp. $\Phi^{t v_-}(p) \in \Sigma_-$ / $\Phi^{t v_0}(p) \in \Sigma$). An *inaccessible* switching point is a switching point that is not accessible;
2. An *upper* (resp. *lower*/*centrally*) *escapable* switching point is a switching point $p \in \Sigma$ for which there exists $\varepsilon \in \mathbb{R}_{>0}$ such that, for every $t \in \left]0, \varepsilon\right[$, $\Phi^{t v_+}(p) \in \Sigma_+$ (resp. $\Phi^{t v_-}(p) \in \Sigma_-$ / $\Phi^{t v_0}(p) \in \Sigma$). An *inescapable* switching point is a switching point that is not escapable.

Transversality of double tangencies In the following definition, $\sphericalangle(v_+(p), v_-(p))$ is the angle between the vectors $v_+(p)$ and $v_-(p)$.

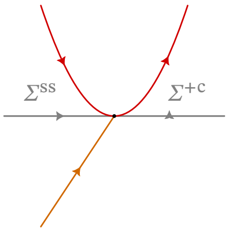
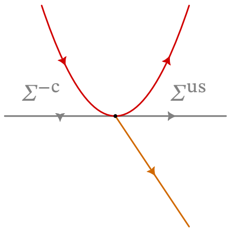
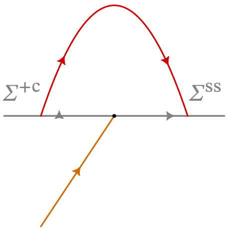
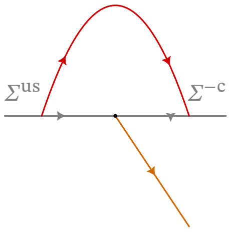
Simple fold	Accessible	Escapable
Visible		
Invisible		

Table 2. Simple fold tangency points.

Definition 2.9. Let $\boldsymbol{M}$ be a $\mathscr{C}^{m+1}$-manifold, Σ a switching submanifold of $\boldsymbol{M}$ and v a Σ-piecewise $\mathscr{C}^{m+1}$-smooth vector field on $\boldsymbol{M}$. A double tangency point $p \in \Sigma^{dt}$ is

1. *cooriented* if $\sphericalangle(v_+(p), v_-(p)) = 0$.
2. *transverse* if $0 < \sphericalangle(v_+(p), v_-(p)) < \pi$.
3. *contraoriented* if $\sphericalangle(v_+(p), v_-(p)) = \pi$.

Notice that transverse double tangencies $p \in \Sigma^{dt}$ can only occur when $\dim(M) \geq 3$ because otherwise $\dim(\Sigma) \leq 1$, so the vector fields $v_+(p)$ and $v_-(p)$ have to be parallel.

2.1.5 Topological properties of the regions of the switching manifold

Proposition 2.2. *Let $\boldsymbol{M}$ be a $\mathscr{C}^{m+1}$-manifold, Σ a switching submanifold of $\boldsymbol{M}$ and v a Σ-piecewise $\mathscr{C}^{m+1}$-smooth vector field on $\boldsymbol{M}$.*

1. *the upper and lower crossing regions Σ^{+c} and Σ^{-c} and the stable and unstable sliding regions Σ^{ss} and Σ^{us} are relatively open in Σ;*
2. *the tangency region Σ^{t} is relatively closed in Σ and is the boundary $\partial(\Sigma^{c} \cup \Sigma^{s})$ of the crossing and sliding regions in Σ;*
3. *the simple tangency region Σ^{st} is relatively open in Σ^{t};*
4. *the double tangency region Σ^{dt} is relatively closed in Σ^{t} and is the boundary $\partial\Sigma^{st}$ of the single tangency region in Σ^{t}.*

2.2 Structural stability

2.3 Sliding vector field

We are now going to define a vector field on the sliding manifold Σ^{s}, called the sliding vector field v_0. Filippov's convention stablishes that this vector field, at a tangency point $p \in \Sigma^{s}$, must be the convex combination of the vectors $v_+(p)$ and $v_-(p)$ which is tangent to to Σ^{s}. Notice that, since $p \in \Sigma^{s}$, then $\partial_{v_+}h(p)\partial_{v_-}h(p) \leq 0$, so $\partial_{v_+}h(p)$ and $\partial_{v_-}h(p)$ have oppostive signs, which means that $v_+(p)$ and $v_-(p)$

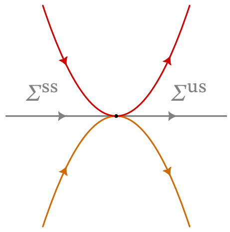
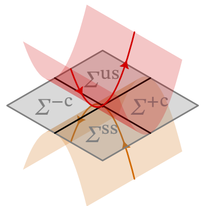
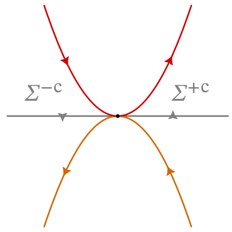
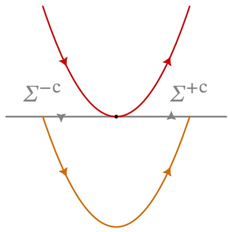
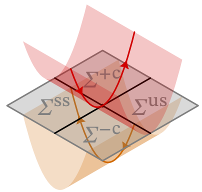
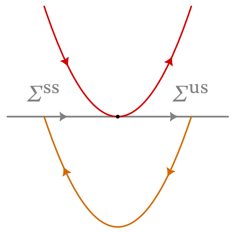
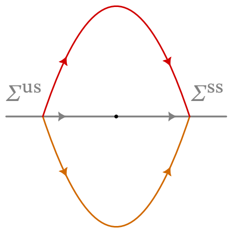
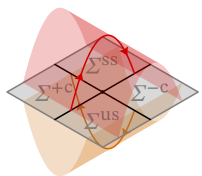
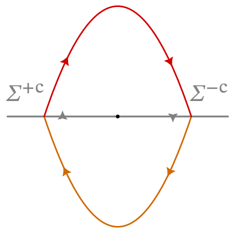
Double fold	Cooriented	Transverse	Contraoriented
Visible			
Visible-invisible			
Invisible			

Table 3. Transversality of double fold tangency points.

point to opposite sides of Σ^{s}, hence there is always a convex combination of them that is tangent to Σ^{s}. Let us calculate explicitly the value of the sliding vector field v_0 relative to the upper and lower vector fields v_+ and v_-.

Proposition 2.3. *Let $\boldsymbol{M}$ be a $\mathscr{C}^{m+1}$-manifold, Σ a switching submanifold of $\boldsymbol{M}$, v a Σ-piecewise $\mathscr{C}^{m+1}$-smooth vector field on $\boldsymbol{M}$ and $p \in \Sigma^{\mathrm{s}}$ a sliding point. The convex combination of the vectors $v_+(p)$ and $v_-(p)$ that is tangent to Σ at p is*

$$v_0(p) := \frac{-\partial_{v_-} h(p)}{\partial_{v_+} h(p) - \partial_{v_-} h(p)} v_+(p) + \frac{\partial_{v_+} h(p)}{\partial_{v_+} h(p) - \partial_{v_-} h(p)} v_-(p).$$

Proof. A convex combination of $v_+(p)$ and $v_-(p)$ is given, for some $s \in [0,1]$, by

$$c_s(p) := (1-s)v_+(p) + s v_-(p).$$

If this convex combination is tangent to Σ, it must satisfy $0 = \langle \nabla h(p), c_s(p) \rangle = \partial_{c_s(p)} h(p)$, so by the $\mathbb{R}$-linearity of the flow derivative ∂ in the vector field argument,

$$0 = \partial_{c_s(p)} h(p) = (1-s)\partial_{v_+} h(p) + s\partial_{v_-} h(p) = \partial_{v_+} h(p) + s(\partial_{v_-} h(p) - \partial_{v_+} h(p)).$$

Since $p \in \Sigma^{\mathrm{s}}$, then $\partial_{v_+} h(p) \partial_{v_-} h(p) < 0$, hence $\partial_{v_+} h(p) - \partial_{v_-} h(p) \neq 0$, so it follows that

$$s = \frac{\partial_{v_+} h(p)}{\partial_{v_+} h(p) - \partial_{v_-} h(p)}.$$

If $\partial_{v_+}h(p) > 0$, then $\partial_{v_-}h(p) < 0$, so $\partial_{v_+}h(p) - \partial_{v_-}h(p) > 0$ and $\partial_{v_+}h(p) < \partial_{v_+}h(p) - \partial_{v_-}h(p)$. From this we obtain

$$0 < \frac{\partial_{v_+}h(p)}{\partial_{v_+}h(p) - \partial_{v_-}h(p)} < 1.$$

If $\partial_{v_+}h(p) < 0$, then $\partial_{v_-}h(p) > 0$, so $\partial_{v_+}h(p) - \partial_{v_-}h(p) < 0$ and $\partial_{v_+}h(p) > \partial_{v_+}h(p) - \partial_{v_-}h(p)$. From this we obtain

$$0 < \frac{\partial_{v_+}h(p)}{\partial_{v_+}h(p) - \partial_{v_-}h(p)} < 1.$$

In both cases, $0 < s < 1$. For $p \in \Sigma^{\mathrm{st}}$, either $\partial_{v_+}h(p) = 0$ or $\partial_{v_-}h(p) = 0$, but not both, so $\partial_{v_+}h(p) - \partial_{v_-}h(p) \neq 0$ and we obtain $s = 0$ or $s = 1$, respectively. We conclude that

$$c_s(p) = (1-s)v_+(p) + sv_-(p) = \frac{-\partial_{v_-}h(p)}{\partial_{v_+}h(p) - \partial_{v_-}h(p)}v_+(p) + \frac{\partial_{v_+}h(p)}{\partial_{v_+}h(p) - \partial_{v_-}h(p)}v_-(p). \qquad \blacksquare$$

Notice that, for a sliding point $p \in \Sigma^{\mathrm{s}}$, it holds that $\partial_{v_+}h(p)\partial_{v_-}h(p) < 0$, so $\partial_{v_+}h(p)$ and $\partial_{v_-}h(p)$ have different signs and therefore $\partial_{v_+}h(p) - \partial_{v_-}h(p) \neq 0$. For simple tangency points, it holds that either $\partial_{v_+}h(p) \neq 0$ and $\partial_{v_-}h(p) = 0$, or $\partial_{v_+}h(p) = 0$ and $\partial_{v_-}h(p) \neq 0$, therefore we also obtain $\partial_{v_+}h(p) - \partial_{v_-}h(p) \neq 0$.

Definition 2.10. Let $\boldsymbol{M}$ be a $\mathscr{C}^{m+1}$-manifold, Σ a switching submanifold of $\boldsymbol{M}$ and v a Σ-piecewise $\mathscr{C}^{m+1}$-smooth vector field on $\boldsymbol{M}$. The *sliding vector field* on $\Sigma^{\mathrm{s}} \cup \Sigma^{\mathrm{st}}$ is the vector field $v_0 : \Sigma^{\mathrm{s}} \cup \Sigma^{\mathrm{st}} \to \mathrm{T}(\Sigma^{\mathrm{s}} \cup \Sigma^{\mathrm{st}})$ defined for each $p \in \Sigma^{\mathrm{s}} \cup \Sigma^{\mathrm{st}}$ by

$$v_0(p) := \frac{-\partial_{v_-}h(p)}{\partial_{v_+}h(p) - \partial_{v_-}h(p)}v_+(p) + \frac{\partial_{v_+}h(p)}{\partial_{v_+}h(p) - \partial_{v_-}h(p)}v_-(p).$$

Question 2.3. What happens to the sliding vector field v_0 near a singularity of the upper or lower vector fields that lies on Σ^{t}?

Proposition 2.4. *Let $\boldsymbol{M}$ be a $\mathscr{C}^{m+1}$-manifold, Σ a switching submanifold of $\boldsymbol{M}$ and v a Σ-piecewise $\mathscr{C}^{m+1}$-smooth vector field on $\boldsymbol{M}$. The sliding vector field $v_0 : \Sigma^{\mathrm{s}} \to \mathrm{T}\Sigma^{\mathrm{s}}$ is $\mathscr{C}^m$-smooth.*

Proof. This is a direct consequence of the definition of v_0 since $\partial_{v_+}h$ and $\partial_{v_-}h$ are $\mathscr{C}^m$-smooth scalar fields. $\blacksquare$

Definition 2.11. Let $\boldsymbol{M}$ be a $\mathscr{C}^{m+1}$-manifold, Σ a switching submanifold of $\boldsymbol{M}$ and v a Σ-piecewise $\mathscr{C}^{m+1}$-smooth vector field on $\boldsymbol{M}$. A *pseudo-equilibrium* of v is an equilibrium of the sliding vector field v_0: a point $p \in \Sigma^{\mathrm{s}} \cup \Sigma^{\mathrm{st}}$ such that $v_0(p) = 0$.

2.4 Piecewise smooth orbits

Before we define piecewise smooth orbits that are integral solutions of a piecewise smooth vector field, we just fix terminoly and notation for the concatenation of two trajectories. For that, we denote the *lower* and *upper extremes* of an interval I ⊆ $\mathbb{R}$ by $\partial_- I := \min \partial I$ and $\partial_+ I := \max \partial I$, respectively.

Definition 2.12. Let $\mathbf{X}$ be a (Hausdorff) topological space, $x \in X$ a point, $I, I' \subseteq \mathbb{R}$ intervals with $t_x := \partial_+ I = \partial_- I'$, and $\gamma : I \to X$ and $\gamma' : X \to$ continuous paths such that

$$x = \lim_{t \nearrow \partial_+ I} \gamma(t) = \lim_{t \searrow \partial_- I'} \gamma'(t).$$

The *concatenation* of γ with γ' at p is the trajectory

$$\gamma \cup \gamma' : I \cup \{t_x\} \cup I' \longrightarrow X$$

$$t \longmapsto \gamma \cup \gamma'(t) := \begin{cases} \gamma(t) & t \in I \\ x & t = t_x \\ \gamma'(t) & t \in I'. \end{cases}$$

Proposition 2.5. *Let* $\mathbf{X}$ *be a (Hausdorff) topological space,* $x \in X$ *a point,* $I, I' \subseteq \mathbb{R}$ *intervals with* $t_x := \partial_+ I = \partial_- I'$*, and* $\gamma : I \to X$ *and* $\gamma' : X \to$ *continuous paths such that*

$$x = \lim_{t \nearrow \partial_+ I} \gamma(t) = \lim_{t \searrow \partial_- I'} \gamma'(t).$$

The concatenation $\gamma \cup \gamma' : I \cup \{t_x\} \cup I' \to X$ *is a continuous path.*

Definition 2.13. Let $\boldsymbol{M}$ be a $\mathscr{C}^{m+1}$-manifold, Σ a switching submanifold of $\boldsymbol{M}$ and v a Σ-piecewise $\mathscr{C}^{m+1}$-smooth vector field on $\boldsymbol{M}$. A *piecewise smooth orbit* of v is a trajectory $\gamma : I \to M$ that satisfies the following local conditions: for every $t_0 \in I$ and $p := \gamma(t_0)$, there exists $\varepsilon \in \mathbb{R}_{>0}$ such that

1. (Regular orbit) If $p \in M \setminus \Sigma$, then, for every $t \in]-\varepsilon, \varepsilon[$, $\gamma(t_0 + t) \in M \setminus \Sigma$ and
$$\gamma(t_0 + t) = \Phi^{tv}(p).$$
That is, γ is locally the flow of v.
2. (Crossing orbit) If $p \in \Sigma^{\mathrm{c}}$, then, for every $t \in]-\varepsilon, \varepsilon[\setminus \{0\}$, $\gamma(t_0 + t) \in M \setminus \Sigma$ and
$$\gamma(t_0 + t) = \Phi^{tv}(p).$$
That is, γ is locally a concatenation at p of the flow of v on the upper and lower regions of Σ.
3. (Sliding orbit) If $p \in \Sigma^{\mathrm{s}}$, then one of the following options hold:
   - 3.1. (Regular sliding orbit) For every $t \in]-\varepsilon, \varepsilon[$, $\gamma(t_0 + t) \in \Sigma^{\mathrm{s}}$ and
   $$\gamma(t_0 + t) = \Phi^{tv_0}(p).$$
   That is, γ is locally the flow of v_0.
   - 3.2. (Superior escaping sliding orbit) For every $t \in]-\varepsilon, 0]$, $\gamma(t_0 + t) \in \Sigma^{\mathrm{us}}$, for every $t \in]0, \varepsilon[$, $\gamma(t_0 + t) \in M \setminus \Sigma$, and
   $$\gamma(t_0 + t) = \begin{cases} \Phi^{tv_0}(p), & t < 0 \\ \Phi^{tv_+}(p) & t \geq 0 \end{cases}$$
   That is, γ is locally a concatenation at p of the flows of v_0 and v_+.
   - 3.3. (Inferior escaping sliding orbit) For every $t \in]-\varepsilon, 0]$, $\gamma(t_0 + t) \in \Sigma^{\mathrm{us}}$, for every $t \in]0, \varepsilon[$, $\gamma(t_0 + t) \in M \setminus \Sigma$, and
   $$\gamma(t_0 + t) = \begin{cases} \Phi^{tv_0}(p), & t < 0 \\ \Phi^{tv_-}(p) & t \geq 0 \end{cases}$$
   That is, γ is locally a concatenation at p of the flows of v_0 and v_-.
   - 3.4. (Superior accessing sliding orbit) For every $t \in]-\varepsilon, 0[$, $\gamma(t_0 + t) \in M \setminus \Sigma$, for every $t \in [0, \varepsilon[$, $\gamma(t_0 + t) \in \Sigma^{\mathrm{ss}}$, and
   $$\gamma(t_0 + t) = \begin{cases} \Phi^{tv_+}(p), & t \leq 0 \\ \Phi^{tv_0}(p) & t > 0 \end{cases}$$
   That is, γ is locally a concatenation at p of the flows of v_+ and v_0.

3.5. (Inferior accesing sliding orbit) For every $t \in \,]-\varepsilon, 0[$, $\gamma(t_0 + t) \in M \smallsetminus \Sigma$, for every $t \in [0, \varepsilon[$, $\gamma(t_0 + t) \in \Sigma^{\mathrm{ss}}$, and

$$\gamma(t_0 + t) = \begin{cases} \Phi^{t v_+}(p), & t \leq 0 \\ \Phi^{t v_0}(p) & t > 0 \end{cases}$$

That is, γ is locally a concatenation at p of the flows of v_- and v_0.

4. (Tangency point) If $p \in \Sigma^{\mathrm{t}}$, the orbit is a concatenation at p of an orbit of v_+, v_- or v_0 — if p is upper, lower or centrally accessible, respectively — or of a sationary orbit — if p is inaccessible — with an orbit of v_+, v_- or v_0 — if p is upper, lower or centrally escapable, respectively — or with a sationary orbit — if p is inescapable.

A *maximal piecewise smooth orbit* of v is a piecewise smooth orbit of v that cannot be extended to a strictly larger interval domain.

This means that a piecewise smooth orbit of v is a continuous piecewise $\mathscr{C}^{m+1}$-smooth trajectory on M, with points of non-smoothness only on the switching manifold Σ.

Question 2.4. What are the types of tangency points and concatenations that can occur together?

Coding piecewise smooth orbits As a consequence of our definition of piecewise smooth orbits for the piecewise smooth vector field v, we do not have uniqueness of solutions, neither for positive nor for negative time. This is so because Filippov's convention establishes that the piecewise vector field v should be understood as been set valued on the switching manifold Σ, and for some particular points on Σ we can define the sliding vector field.

Nontheless, we still have some regularity, some control over the behavior of orbits Besides the points of concatenation of regular orbits, the orbit is uniquely defined by the flows of the upper vector field v_+, the lower vector field v_- or the sliding vector v_0. Definition 2.13 implies that for each piecewise smooth orbit γ of v defined on an interval $I \subseteq \mathbb{R}$ (we assume $0 \in I$ for simplicity), there exists a sequence (q_i, σ_i) of switching points $q_i \in \Sigma$ and signs $\sigma_i \in \{1, -1\}$ that indicates the points of concatenation and sides of Σ this concatenation occurs.

These points can be crossing, sliding, or tangency points. For our purposes, we ignore crossing points, since the orbit does not loose uniqueness at them. We will focus on sliding points, and tangency points on their closure. Especifically, we distinguish accessing points Σ^{ss} and escaping points Σ^{us} because the orbit looses uniqueness at escaping points when going forward, and at accessing points when going backward. Precisely, we state proposition 2.6, but first we define some terms that will ease its formulation.

Definition 2.14. Let $\boldsymbol{M}$ be a $\mathscr{C}^{m+1}$-manifold, Σ a switching submanifold of $\boldsymbol{M}$ and v a Σ-piecewise $\mathscr{C}^{m+1}$-smooth vector field on $\boldsymbol{M}$. Consider a piecewise smooth orbit $\gamma\colon I \to M$ of v, $\tau \in I^\circ$, and $\sigma \in \{1, -1\}$. We say

1. γ *accesses* Σ^{ss} *from* Σ_σ through the point $a \in (\Sigma^{\mathrm{ss}})^\bullet$ at time τ when $\gamma(\tau) = a$ and there exists $\varepsilon \in \mathbb{R}_{>0}$ such that, for every $t \in \,]\tau - \varepsilon, \tau[$, $\gamma(t) \in \Sigma_\sigma$;
2. γ *escapes* Σ^{us} *to* Σ_σ through the point $e \in (\Sigma^{\mathrm{us}})^\bullet$ at time τ when $\gamma(\tau) = e$ and there exists $\varepsilon \in \mathbb{R}_{>0}$ such that, for every $t \in \,]\tau, \tau + \varepsilon[$, $\gamma(t) \in \Sigma_\sigma$.

Proposition 2.6. *Let* $\boldsymbol{M}$ *be a* $\mathscr{C}^{m+1}$*-manifold,* Σ *a switching submanifold of* $\boldsymbol{M}$ *and* v *a* Σ*-piecewise* $\mathscr{C}^{m+1}$*-smooth vector field on* $\boldsymbol{M}$*. For each piecewise smooth orbit* $\gamma\colon I \to M$ *of* v *(assume that* $0 \in I$ *for simplicity),*

1. *there exist a discrete interval $I_a \subseteq \mathbb{Z}$, a sequence $a\colon\ I_a \to (\Sigma^{\mathrm{ss}})^\bullet$ of accessing sliding points (or tangency points in their closure), a sequence $\sigma_a(\cdot)\colon\ I_a \to \{1,-1\}$ of signs, and a strictly increasing sequence $\tau_a(\cdot)\colon\ I_a \to I$ of times which satisfy that*
   1.1. *for each $i \in I_a$, γ accesses Σ^{ss} from $\Sigma_{\sigma_a(i)}$ through the point $a(i)$ at time $\tau_a(i)$;*
   1.2. *if $i-1 \in I_a$, the last time γ accessed Σ^{ss} before $\tau_a(i)$ was at $\tau_a(i-1)$;*
   1.3. *$-1 \leq \tau_a(0) < 0$ and I_a is maximal with these properties;*
2. *there exist a discrete interval $I_e \subseteq \mathbb{Z}$, a sequence $e\colon\ I_e \to (\Sigma^{\mathrm{us}})^\bullet$ of escaping sliding points (or tangency points in their closure), a sequence $\sigma_e(\cdot)\colon\ I_e \to \{1,-1\}$ of signs, and a strictly increasing sequence $\tau_e(\cdot)\colon\ I_e \to I$ of times which satisfy that*
   2.1. *for each $i \in I_e$, γ escapes Σ^{us} to $\Sigma_{\sigma_e(i)}$ through the point $e(i)$ at time $\tau_e(i)$;*
   2.2. *if $i+1 \in I_a$, the first time γ escapes Σ^{us} after $\tau_a(i)$ is at $\tau_a(i+1)$;*
   2.3. *$-1 < \tau_e(0) \leq 0$ and I_e is maximal with these properties.*

3 Topological dynamics of piecewise smooth vector fields

3.1 Topological transitivity

4 Orbit spaces

4.1 Orbit space and orbit space flow

Definition 4.1. Let $\boldsymbol{M}$ be a $\mathscr{C}^{m+1}$-manifold, Σ a switching submanifold of $\boldsymbol{M}$ and v a Σ-piecewise $\mathscr{C}^{m+1}$-smooth vector field on $\boldsymbol{M}$. The *orbit space* of v is the set of all maximal piecewise smooth orbits of v:

$$\tilde{M} := \{\gamma\colon\ I_\gamma \to M \mid \gamma \text{ is a maximal piecewise smooth orbit of } v\}.$$

Notice that the orbit space $\tilde{M}$ does not depend only on the manifold $\boldsymbol{M}$, but also on the piecewise smooth vector field v, so this is an abuse of notation. We do this because the notation for the sets M and $\tilde{M}$ resemble each other.

Definition 4.2. Let $\boldsymbol{M}$ be a $\mathscr{C}^{m+1}$-manifold, Σ a switching submanifold of $\boldsymbol{M}$ and v a piecewise smooth vector field on $\boldsymbol{M}$. The *flow domain* on $\tilde{M}$ is the set

$$D_{\tilde{\Phi}} := \bigcup_{\gamma\in\tilde{M}} I_\gamma \times \{\gamma\} \subseteq \mathbb{R}\times\tilde{M}.$$

The *flow on $\tilde{M}$ induced by v* is the transformation

$$\begin{aligned} \tilde{\Phi}\colon\ D_{\tilde{\Phi}} &\longrightarrow \tilde{M} \\ (t,\gamma) &\longmapsto \tilde{\Phi}^t(\gamma)\colon\ I_\gamma - t \longrightarrow M \\ &\qquad\qquad\qquad t' \longmapsto \gamma(t+t'). \end{aligned}$$

When every maximal piecewise smooth orbit $\gamma \in \tilde{M}$ is defined on $\mathbb{R}$, this simplifies to the flow

$$\begin{aligned} \tilde{\Phi}\colon\ \mathbb{R}\times\tilde{M} &\longrightarrow \tilde{M} \\ (t,\gamma) &\longmapsto \tilde{\Phi}^t(\gamma)\colon\ \mathbb{R} \longrightarrow M \\ &\qquad\qquad\qquad t' \longmapsto \gamma(t+t'). \end{aligned}$$

We will show that this is a continuous flow after we introduce a topology on $\tilde{M}$.

4.2 Metric space structure of the orbit space

We now take in the manifold $\boldsymbol{M}$ a compatible distance function $|\cdot,\cdot|$ induced by a Riemannian metric tensor $\langle\cdot,\cdot\rangle$. We will also assume that the the piecewise smooth vector field v is bounded, that is, its *supremum norm* is finite:

$$\|v\| := \sup_{p\in M}|v(p)| < \infty.$$

This implies that the sliding vector field v_0 is also bounded. We also assume $\|v\| > 0$. We may ocasionally assume that v is Lipschitz continuous instead of bounded. This menas that its *dilation* is finite:

$$\langle\langle v\rangle\rangle := \sup_{p\neq p'\in M}\frac{|v(p')-v(p)|}{|p,p'|} < \infty.$$

Definition 4.3. Let $\boldsymbol{M}$ be a $\mathscr{C}^{m+1}$-manifold with Riemannian distance $|\cdot,\cdot|$, Σ a switching submanifold of $\boldsymbol{M}$ and v a bounded piecewise smooth vector field on $\boldsymbol{M}$. The (*supremum*) *distance* on $\tilde{M}$ induced by $|\cdot,\cdot|$ is the function

$$\begin{aligned}\|\cdot,\cdot\|:\ &\tilde{M}\times\tilde{M}\longrightarrow\mathbb{R}\\ &(\gamma_0,\gamma_1)\longmapsto\|\gamma_0,\gamma_1\| := \sum_{i\in\mathbb{Z}}2^{-|i|}\sup_{i\leq t<i+1}|\gamma_0(t),\gamma_1(t)|.\end{aligned}$$

4.3 Topological transitivity, above and below

4.4 Topological entropy