Topological Dynamics

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1 Preliminaries

A *transformation* (or *function*) f from a *set* C to a set C' is denoted $f : C \to C'$ and the composition os transformations $f : C \to C'$ and $f' : C' \to C''$ is denoted $f' \circ f : C \to C''$. The *identity* transformation in C is $\mathrm{I} = \mathrm{I}_C : C \to C$.

The *empty set* is denoted $\mathbb{0}$, the *point set*, $\mathbb{1} = \{0\}$, and the *double set*, $\mathbb{2} = \{0, 1\}$. Given a "universe" set X, the *power set* of X (the set of all its subsets) is denoted 2^X, and the *set complement* (*in* X) of a subset $S \subseteq X$ is denoted $\overline{S} = X \setminus S$.

The *natural* numbers are denoted $\mathbb{N}$ and include 0 (zero), and, for every natural number $n \in \mathbb{N}$, the set of the first n natural numbers is denoted $[\![n]\!] := \{n' \in \mathbb{N} \mid n' < n\} = \{0, \dots, n-1\}$. The *integers* are denoted $\mathbb{Z}$, the *rational* numbers, $\mathbb{Q}$, and the *real* numbers, $\mathbb{R}$. The *positive*, *strictly positive*, *negative* and *strictly negative* integers are denoted $\mathbb{Z}_{\geq 0}$, $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\leq 0}$ and $\mathbb{Z}_{<0}$, respectively. Analogous notation is used for other ordered number sets. We denote the *infimum* of a set S by $\inf S$ and its supremum by $\sup S$.

1.1 Topological spaces

We denote a *topological space* by $\boldsymbol{X} = (X, \mathcal{T})$, being X the set of points of the space and $\mathcal{T}$ its topology[1]. The *trivial* topology on a set X is $\{\mathbb{0}, X\}$ and the *discrete* topology on X is the power set 2^X. We denote the *topological interior* of a set $S \subseteq X$ by S° and its *topological closure* by $S^\bullet$. We state some basic definitions and propositions for later reference. Most of the omitted proofs can be found in the book *Topology*, by Munkres [Mun00].

Definition 1.1. Let $\boldsymbol{X}$ be a topological space. A *neighborhood* of a point $x \in X$ is a set $V \subseteq X$ such that $x \in V^\circ$. A *neighborhood* of a set $S \subseteq X$ is a set $V \subseteq X$ such that $S \subseteq V^\circ$.

Definition 1.2. Let $\boldsymbol{X}$ be a topological space and $x \in X$. A *local neighborhood basis* at x is a collection $\mathcal{B} \subseteq 2^X$ of neighborhoods of x such that, for every neighborhood $V \subseteq X$ of x, there exists some $B \in \mathcal{B}$ such that $B \subseteq V$.

Definition 1.3. Let $\boldsymbol{X}$ be a topological space. An *isolated* point is a point $x \in X$ that has a neighborhood without other points of the space (that is, $\{x\}$ is open).

Definition 1.4. A *discrete* topological space is a topological space $\boldsymbol{X}$ in which every point is isolated. Its topology is 2^X, the *discrete* topology on X.

Definition 1.5. A *perfect* (or *accumulated*) topological space is a topological space $\boldsymbol{X}$ that has no isolated points.

1.1.1 Separability

We start by defining several notions subsetes of a topological space can be distinguished. The first notions are not restricted to topological spaces. Let X be a set and $S, S' \subseteq X$ subsetes. They are *distinct* when $S \neq S'$, which is a logical notion that for sets is equivalent to the existence of an element of one set that is not an element of the other, and they are *disjoint* when $S \cap S' = \mathbb{0}$, which is a set theoretic notion.

Definition 1.6. Let $\boldsymbol{X}$ be a topological space, $x, x' \in X$ points, and $S, S' \subseteq X$ subsets.

1. We generally use boldface to indicate the space/categorial object $\boldsymbol{X}$, including its structure (its topology $\mathcal{T}$, distance $|\cdot, \cdot|$, binary operation $+$, ...), while the normal weight font denotes the underlying set X.

1. The points x and x' are *topologically distinct* when one of them has a neighborhood that does not contain the other.
2. The sets S and S' are *nearly separated* when each set has a neighborhood that is not a neighborhood of the other. (equivalently, when each is disjoint from the closure of the other)
3. The sets S and S' are *separated* when they have disjoint neighborhoods.
4. The sets S and S' are *separated by close neighborhood* when they have disjoint closed neighborhoods.
5. The sets S and S' are *separated by continuous function* when there exists a continuous function $f : X \to \mathbb{R}$ such that $S \subseteq f^{-1}(0)$ and $S' \subseteq f^{-1}(1)$.
6. The sets S and S' are *precisely separated by continuous function* when there exists a continuous function $f : X \to \mathbb{R}$ such that $S = f^{-1}(0)$ and $S' = f^{-1}(1)$.

Each condition in DEFINITION 1.6 is implied by the following condition. In what follows, we will use these definition to enunciate and prove many different properties of topological spaces.

Definition 1.7. A *punctual* (or T_0*-separated*) topological space is a topological space $\boldsymbol{X}$ in which distinct points are topologically distinct.

These are the spaces that can be recovered from abstract "topologies without points".

$\mathbf{T_1}$-separated spaces

Definition 1.8. An *accessible* (or T_1*-separated*) topological space is a topological space $\boldsymbol{X}$ in which distinct points are nearly separated.

An equivalent definition of an accessible space is a topological space in which, for every point $x \in X$, the singleton set $\{x\}$ is closed. This is also equivalent to every finite set being closed.

Proposition 1.1. *A topological space $\boldsymbol{X}$ is accessible if, and only if, every finite set $F \subseteq X$ is closed.*

Lemma 1.2. *Let $\boldsymbol{X}$ be an accessible and perfect topological space.*

1. *There are no non-empty finite open sets $F \subseteq X$.*
2. *If $U \subseteq X$ is a non-empty open set and $F \subseteq X$ is finite, then $U \setminus F$ is a non-empty open set.*

Proof. 1. Let $F \subseteq X$ be a finite open set. Suppose, for the sake of contradiction, that F were non-empty, and take $x \in F$. Notice that the complement of $\{x\}$ satisfies

$$\overline{\{x\}} = X \cap \overline{\{x\}} = (\overline{F} \cup F) \cap \overline{\{x\}} = \overline{F} \cup (F \cap \overline{\{x\}}).$$

Since F is open, its complement $\overline{F}$ is closed. Since F is finite, $F \setminus \{x\} = F \cap \overline{\{x\}}$ would be finite, hence closed because $\boldsymbol{X}$ is accessible (PROPOSITION 1.1). Then it would follow that $\overline{\{x\}}$ is a union of closed sets, thefore closed, so $\{x\}$ would be open. This would contradict the fact that $\boldsymbol{X}$ is perfect, so we conclude that F is empty.

2. Since F is finite and $\boldsymbol{X}$ is accessible, F is closed (LEMMA 1.2), so $U \setminus F$ is open. Besides that, $U \setminus F$ is also non-empty, since otherwise we would have $U \subseteq F$, so the non-empty open set U would be a finite, which cannot happen in an accessible and perfect space (by the previous item). ■

$\mathbf{T_2}$-separated spaces We now define the most important separation property of a topologicl space. At first glance it may just seem like one among many in the hierarchy of separation properties, but it becomes evident throughout the study of topological spaces and many other related areas that it is essential.

Definition 1.9. A *separated* (or T_2*-separated*, or *Hausdorff*) topological space is a topological space $\boldsymbol{X}$ in which distinct points are separated. We also say that $\mathcal{T}$ is a *separated* (*Hausdorff*) topology on X.

Definition 1.10. A topological space $\boldsymbol{X}$ is separated (Hausdorff) if, and only if, limits of nets are unique. (In particular, in separated spaces limits of sequences are unique.)

Proposition 1.3. *A subspace of a separated topological space is separated.*

Proposition 1.4. *Let X be a set.*

1. *The discrete topology 2^X on X is separated.*
2. *If $\mathcal{T}$ is a separated topology on X, then every greater (finer/stronger) topology $\mathcal{T}' \supseteq \mathcal{T}$ on X is separated.*

T_3-separated spaces

Definition 1.11. A *regular* topological space is a topological space $\boldsymbol{X}$ in which every disjoint point set and closed set are separated. A T_3*-separated* space is a regular topological space that is separated (Hausdorff).

T_4-separated spaces

Definition 1.12. A *normal* topological space is a topological space $\boldsymbol{X}$ in which disjoint closed sets are separated. A T_4*-separated* space is a normal topological space that is separated (Hausdorff).

1.1.2 Bases and countability

Definition 1.13. Let $\varkappa$ be a cardinal.

1. A $\varkappa$*-generated* topological space is a topological space $\boldsymbol{X}$ that admits a basis for its topology of cardinality $\varkappa$. The case $\varkappa = \aleph_0$ is also called *second-countable*.
2. A *locally* $\varkappa$*-generated* topological space is a topological space $\boldsymbol{X}$ in which every point admits a local basis of cardinality $\varkappa$. The case $\varkappa = \aleph_0$ is also called *first-countable*.

Notice that every second-countable space is first countable. To converse is not true; for instance, an uncountable discrete space is first-countable but it is not second-coutable.

Definition 1.14. Let $\varkappa$ be a cardinal. A $\varkappa$*-dense* topological space is a topological space that admits a dense set of cardinality $\varkappa$.

The $\aleph_0$-dense spaces are often called "separable" spaces, but we will avoid this terminology because "separation" and related words are overused in topology, so to avoid confusion we give preference to the separation axioms of SECTION 1.1.1.

Proposition 1.5. *An $\aleph_0$-generated topological space $\boldsymbol{X}$ is $\aleph_0$-dense.*

Proof. Let $\mathcal{B}$ be a countable basis of non-empty open sets and take, for each $B \in \mathcal{B}$, a point $p_B \in B$. Then the set $D := \{p_B \mid b \in \mathcal{B}\}$ is dense in $\boldsymbol{X}$. ∎

Proposition 1.6. *Let $\boldsymbol{X}$ be a Hausdorff topological space and $\varkappa$ a cardinal. If $\boldsymbol{X}$ is $\varkappa$-dense, then X has cardinality at most $2^{2^\varkappa}$. If $\boldsymbol{X}$ is first-countable (locally $\aleph_0$-generated), then X has cardinality at most $2^\varkappa$.*

Proof. Check [Wik25b, § Separability versus second countability]. ∎

1.1.3 Connectedness

Definition 1.15. A *connected* topological space is a topological space $\boldsymbol{X}$ that does not admit a non-trivial open partition (or, equivalently, if $A, A' \subseteq X$ are open sets such that $A \cap A' = \mathbb{0}$ and $A \cup A' = X$, then $\{A, A'\} = \{\mathbb{0}, X\}$). A *disconnected* space is a space that is not connected.

Proposition 1.7. *Let $\boldsymbol{X}$ be a topological space and $\boldsymbol{S} \subseteq \boldsymbol{X}$ a topological subspace. The space $\boldsymbol{S}$ is connected if, and only if, there is no pair of non-empty open sets $A, A' \subseteq X$ which are disjoint, $A \cup A' = S$, and neither one contains a limit point of the other.*

Proposition 1.8. *Let $\boldsymbol{X}$ and $\boldsymbol{X}'$ be topological spaces and $f : X \to X'$ a continuous function. If $\boldsymbol{X}$ is connected, then $f(X) \subseteq X'$ is connected.*

Proposition 1.9. *Let $\boldsymbol{X}$ be a topological space and $\mathcal{C} \subseteq 2^X$ a collection of connected sets such that, for every $C, C' \in \mathcal{C}$, $C \cap C' \neq \mathbb{0}$. Then $\bigcup_{C \in \mathcal{C}} C$ is connected.*

Definition 1.16. Let $\boldsymbol{X}$ be a topological space and $x \in X$. The *connected component* of x is the union of every connected set of $\boldsymbol{X}$ that contains x.

Definition 1.17. A *totally disconnected* space is a topological space $\boldsymbol{X}$ for which the connected component of every point $x \in X$ is $\{x\}$.

Proposition 1.10. *Every discrete space is totally disconnected.*

Proposition 1.11 (Brouwer's theorem)**.** *Every non-empty, perfect, compact, Hausdorff space with a (countable?) basis of closed-open sets is homeomorphic to $\mathbb{2}^\omega$.*

Equivalently, every non-empty, perfect, compact, totally disconnected, metrizable topological space is homeomorphic to $\mathbb{2}^\omega$.

1.1.4 Compactness

Closed intervals of the real line $\mathbb{R}$ have the following important properties: 1. any decreasing sequence of non-empty closed subsets of a closed real interval has non-empty intersection, 2. every continuous real function on a closed real interval has a maximum and minimum, 3. every sequence in a closed real interval has a convergent subsequence, and 4. closed real intervals are bounded (and closed).

Compact spaces are topological spaces that share some of these properties. Every continuous real function on a compact space has a maximum and minimum, and any decreasing sequence of non-empty closed sets on a compact space has non-empty intersection. Also, in compact metric spaces, every sequence has a convergent subsequence. The definition we adopt for compact spaces is equivalent to the intersection of closed sets property, but using unions of open sets instead, as will be clear shortly.

Definition 1.18. Let $\boldsymbol{X}$ be a topological sapce. An *open cover* of $\boldsymbol{X}$ is a collection of open sets $\mathcal{C} \subseteq \mathcal{T}$ such that $X = \bigcup_{C \in \mathcal{C}} C$. An *open subcover* of an open cover $\mathcal{C}$ is a collection $\mathcal{S} \subseteq \mathcal{C}$ that is an open cover of $\boldsymbol{X}$.

Definition 1.19. A *compact* topological space is a topological space $\boldsymbol{X}$ in which every open cover $\mathcal{C}$ of $\boldsymbol{X}$ has a finite open subcover $\mathcal{F} \subseteq \mathcal{C}$ of $\boldsymbol{X}$. We also say that $\mathcal{T}$ is a *compact* topology on X.

Using the induced subspace topology, we have the following caracterization of compact subspaces.

Proposition 1.12. *Let $\boldsymbol{X}$ be a topological space and $\boldsymbol{S} \subseteq \boldsymbol{X}$ a subspace. Then $\boldsymbol{S}$ is compact if, and only if, every open cover of S in X has a finite subcover in X.*

Proof. See [Mun00, Lemma 26.1, p. 164]. ■

As we just mentioned, we can characterize compact spaces with closed sets. This is done by using using set complements and the logical contrapositive. The dual definition of compactness (by using set complements) is the following: a compact space is a topological space in whiuch every collection $\mathcal{C}$ of closed sets that has empty intersection has a finite subcollection $\mathcal{F} \subseteq \mathcal{C}$ that has empty intersection. In this form, this property is not immediately obvious, but by taking the contrapositive, we obtain a much more useful result: PROPOSITION 1.13. Before presenting it, we define a property in order to simplify its statement.

Definition 1.20. Let $\boldsymbol{X}$ be a topological space. A collection of subsets with the *finite intersection property* is a collection $\mathcal{C} \subseteq 2^X$ such that, for every finite subset $\mathcal{F} \subseteq C$,

$$\bigcap_{F \in \mathcal{F}} F \neq \emptyset.$$

In particular, a decreasing sequence of closed non-empty subsets $C_0 \supseteq C_1 \supseteq \dots$ has the finite intersection property.

Proposition 1.13. *Let $\boldsymbol{X}$ be a topological space. Then $\boldsymbol{X}$ is compact if, and only if, for every collection of $\mathcal{C}$ of closed subsets with the finite intersection property,*

$$\bigcap_{C \in \mathcal{C}} C \neq \emptyset.$$

In particular, in a compact space a decreasing sequence of non-empty closed subsets $C_0 \supseteq C_1 \supseteq \dots$ has non-empty intersection.

Proof. This basically follows from the definition of a compact space by taking set complements and the contrapositive. See [Mun00, Theorem 26.9, p. 169] for the details. ■

It is easy to show that compactness is preserved by continuous functions, which turns out to be related to the fact that real functions defined on closed real intervals have a maximum and a minimum.

Proposition 1.14. *Let $\boldsymbol{X}$ and $\boldsymbol{X}'$ be topological spaces and $f : X \to X'$ a continuous transformation. If X is compact, then $f(X) \subseteq X'$ is compact.*

Compactness and smallness Compact spaces are, in some sense, “small”. Broadly speaking, compact spaces can be seen as analogues, in the category of topological spaces, to finite sets in the category of sets. This sense of smallness is also shared by closed real intervals because they are bounded, but we will not explore this relation now.

We first relate compactness and finiteness by some immediate properties that generalize the facts that a single point of the real line is a (trivial) closed interval, and that finite unions of closed real intervals also share the properties listed in the beginning of this subsection.

Proposition 1.15. *The point space $\mathbb{1}$ is compact, and every finite union of compact subspaces of a topological space $\boldsymbol{X}$ is compact. Consequently, every finite space is compact.*

In a more general direction, we can show that compact topologies are in some sense small.

Proposition 1.16. *Let X be a set.*

1. *The trivial topology $\{\emptyset, X\}$ on X is compact.*
2. *If $\mathcal{T}$ is a compact topology on X, then every smaller (coarser/weaker) topology $\mathcal{T}' \subseteq \mathcal{T}$ on X is compact.*

Duality of compact and Hausdorff spaces We will compare the "smallness" of compact spaces with the "largeness" of Hausdorff spaces. Although the definitions of Hausdorff spaces and compact spaces seem unrelated, and not even formulated in similar terms, there is some kind of duality between them. The following propositions can be trivially proved from the definitions, but they shed some light on this duality. This section is based on [Tao09a].

One of the useful properties of closed real intervals is that they are (obviously) closed. One way to understand the relation between the concepts of closed and compact subsets is by relating them both in compact spaces and in Hausdorff spaces. The following results show that this relation is dual on these types of spaces.

Proposition 1.17. *Let $\boldsymbol{X}$ be a compact space and $\boldsymbol{S} \subseteq \boldsymbol{X}$ a subspace. If $\boldsymbol{S}$ is closed, then it is compact.*

Proof. Let $F \subseteq X$ be a closed set and $\mathcal{C}$ an open cover of F. Since F is closed, its complement $\overline{F}$ is open, so $\mathcal{C}' := \mathcal{C} \cup \{\overline{F}\}$ is an open cover of X. Since X is compact, there exists an open subcover $\mathcal{S}' \subseteq \mathcal{C}'$ of X. Defining $\mathcal{S} := \mathcal{S}' \setminus \{\overline{F}\}$, it follows that $\mathcal{S} \subseteq \mathcal{C}$ is an open subcover of F, which proves that F is compact. ■

Proposition 1.18. *Let $\boldsymbol{X}$ be a Hausdorff space and $\boldsymbol{S} \subseteq \boldsymbol{X}$ a subspace. If $\boldsymbol{S}$ is compact, then it is closed.*

Proof. See [Mun00, Theorem 26.3, p. 165]. ■

Now notice that PROPOSITION 1.16 shows that the trivial topology is always compact, and that any topology that is smaller then a compact topology is also compact. Dually, PROPOSITION 1.4 shows that the discrete topology is Hausdorff, and that any topology that is greater then a Hausforff topology is also Hausrdorff. PROPOSITIONS 1.4 and 1.16 can be interpreted as meaning that compact spaces have weaker topologies, while Hausdorff spaces have stronger topologies. Their relation can be better understood by the next proposition.

Proposition 1.19. *Let X be a set, and $\mathcal{T}$ and $\mathcal{T}'$ topologies on X such that $\mathcal{T} \subseteq \mathcal{T}'$. If $(X, \mathcal{T})$ is Hausdorff and $(X, \mathcal{T}')$ is compact, then $\mathcal{T} = \mathcal{T}'$.*

Proof. Since $\mathcal{T} \subseteq \mathcal{T}'$, every closed set in $(X, \mathcal{T})$ is a closed in $(X, \mathcal{T}')$ and every compact set in $(X, \mathcal{T}')$ is a compact in $(X, \mathcal{T})$. PROPOSITION 1.17 implies that every closed set of $(X, \mathcal{T}')$ is compact in $(X, \mathcal{T}')$, and PROPOSITION 1.18 implies that every compact set of $(X, \mathcal{T})$ is closed in $(X, \mathcal{T})$. This shows that $(X, \mathcal{T})$ and $(X, \mathcal{T}')$ have the same closed sets, hence the same open sets, therefore $\mathcal{T} = \mathcal{T}'$. ■

In particular this implies the following.

Proposition 1.20. *Let $\boldsymbol{X}$ be a compact topological space, $\boldsymbol{X}'$ a Hausdorff topological space, and $f : X \to X'$ a continuous transformation. If f is invertible, then f^{-1} is continuous.*

Proof. Let $F \subseteq X$ be a closed set. We will show that $(f^{-1})^{-1}(F)$ is closed. The set F is compact because X is compact (PROPOSITION 1.17), so $f(F) \subseteq f(X) = X'$ is compact because f is continuous, which implis that $f(F)$ is closed because X' is Hausdorff (PROPOSITION 1.18). Since f is invertible, we have $(f^{-1})^{-1}(F) = F$, so we conclude that $(f^{-1})^{-1}(F)$ is closed. ■

Proposition 1.21. *Let $\boldsymbol{X}$ be a Hausdorff topological space and $(K_i)_{i \in I}$ a collection of compact subsets of X. Then $K_\cap := \bigcap_{i \in I} K_i$ is a compact set.*

Proof. Let $\mathcal{C}$ be an open cover of $K_\cap$. We must find a finite subcover of $\mathcal{C}$ that covers $K_\cap$. Since $\boldsymbol{X}$ is Hausdorff, for every $i \in I$ the compact set K_i is closed (PROPOSITION 1.18), so the intersection $K_\cap$ is closed. Take $i_0 \in I$. Then $K_\cap$ is a closed subset of the compact set K_{i_0}, therefore it is also compact. ■

Proposition 1.22. *Let $\boldsymbol{X}$ be a non-empty compact Hausdorff topological space. If $\boldsymbol{X}$ is perfect, then X is uncountable.*

Proof. See [Mun00, Theorem 27.7, p. 176]. ■

Compact spaces and bases

Product of compact spaces

Proposition 1.23. *A product of compact spaces is compact.*

1.1.5 Local compactness

Definition 1.21. A *locally compact* topological space is a topological space in which every point has a compact neighborhood.

Proposition 1.24. *A Hausdorff topological space $\boldsymbol{X}$ is locally compact if, and only if, every point of X has a local base of closed compact neighborhoods.*

Proof. See [Mun00, Theorem 29.2, p. 185]. ■

Proposition 1.25. *Let $\boldsymbol{X}$ be a topological space. Then $\boldsymbol{X}$ is locally compact and Hausdorff if, and only if, there exists a compact Hausdorff superspace $\boldsymbol{X}' \supseteq \boldsymbol{X}$ such that the difference $X' \setminus X$ is a single point. If $\boldsymbol{X}'$ and $\boldsymbol{X}''$ are two such superspaces of $\boldsymbol{X}$, there exists a homeomorphism from $\boldsymbol{X}'$ to $\boldsymbol{X}''$ that is the identity on X.*

If $\boldsymbol{X}$ is compact, then such space $\boldsymbol{X}'$ is the disjoint union of $\boldsymbol{X}$ with the point space $\mathbb{1}$. Otherwise, the point of $X' \setminus X$ is a limit point of X in $\boldsymbol{X}'$.

Proof. See [Mun00, Theorem 29.1, p. 183]. ■

Definition 1.22. Let $\boldsymbol{X}$ be a locally compact Hausdorff space that is not compact. The *one-point compactification* of $\boldsymbol{X}$ is the unique (up to homeomorphism) compact Hausdorff space superspace $\mathfrak{C}(\boldsymbol{X}) \supseteq \boldsymbol{X}$ such that the difference $\mathfrak{C}(\boldsymbol{X}) \setminus X$ is a single point. The *point at infinity* of $\boldsymbol{X}$ is the unique point $\infty_X \in \mathfrak{C}(\boldsymbol{X}) \setminus X$.

The one-point compactification of the d-dimensional real space $\mathbb{R}^d$ is the d-dimensional real sphere $\mathbb{S}^d$, that is,

$$\mathfrak{C}(\mathbb{R}^d) = \mathbb{S}^d$$

Convergence/divergence at infinity

Definition 1.23. Let $\boldsymbol{X}$ be a non-compact, locally compact Hausdorff space. A sequence that *diverges to infinity* in $\boldsymbol{X}$ is a sequence $(x_n)_{n\in\mathbb{N}}$ in X which satisfies that, for every compact set $K \subseteq X$, there exists a natural number $n_K \in \mathbb{N}$ such that, for every natural number $n \in \mathbb{N}$, if $n \geq n_K$, then $x_n \notin K$.

Proposition 1.26. *Let $\boldsymbol{X}$ be a non-compact, locally compact Hausdorff space, $\mathfrak{C}(\boldsymbol{X})$ its one-point compactification, and $(x_n)_{n\in\mathbb{N}}$ a sequence in X. Then $(x_n)_{n\in\mathbb{N}}$ diveges to infinity in $\boldsymbol{X}$ if, and only if, it converges to ∞_X in $\mathfrak{C}(\boldsymbol{X})$.*

This shows that we can use the notation $\lim_{n\to\infty} x_n = \infty_X$ without ambiguity for a sequence that diverges to infinity in $\boldsymbol{X}$.

Definition 1.24. Let $\boldsymbol{X}$ be a non-compact, locally compact Hausdorff space and $z \in \mathbb{C}$. A complex-valued function on X that *equals z at infinity* is a function $f : X \to \mathbb{C}$ that satisfies the following: for every $\varepsilon \in \mathbb{R}_{>0}$, there exists a compact set $K_\varepsilon \subseteq X$ such that, for all $x \in X$, if $x \notin K_\varepsilon$, then $|f(x) - z| < \varepsilon$.

A complex-valued function on X that *vanishes at infinity* is a function $f : X \to \mathbb{C}$ that equals 0 at infinity. The space of all continuous complex-valued functions that vanish at infinity is denoted $\mathscr{C}_0(X, \mathbb{C})$.

Proposition 1.27. *Let $\boldsymbol{X}$ be a locally compact Hausdorff space, $f : X \to \mathbb{C}$ a complex-valued function on X and $z \in \mathbb{C}$. Then f equals z at infinity if, and only if, it can be extended to a function $\mathfrak{E}(f) : \mathfrak{E}(X) \to \mathbb{C}$ on the one-point compactification of $\boldsymbol{X}$ that is continuous at the point at infinity ∞_X and $\mathfrak{E}(f)(\infty_X) = z$.*

These concepts allow connecting locally compact Hausdorff spaces with commutative complete normed involutive algebras over the complex numbers (commutative C*-algebras). The space $\mathscr{C}_0(X, \mathbb{C})$ of complex-value functions that vanish at infinity is a commutative C*-algebra (a sub-algebra of the bounded and continuous complex-value functions), and it is unital if, and only if, $\boldsymbol{X}$ is compact, in which case it equal the space of continuous functions. Chech [MSE10; Wik25a] for further information.

1.2 Metric spaces

We denote a *metric space* by $\boldsymbol{M} = (M, |\cdot,\cdot|)$, being M the set of points of the space and $|\cdot,\cdot| : M \times M \to \mathbb{R}_{\geq 0}$ the distance function of the space, denoted on points $p, p' \in M$ by $|p, p'|$. We denote the *diameter* of a set $S \subseteq M$ by $\Theta(S) := \sup_{p,p' \in S} |p, p'|$ and the *open ball* and *closed ball* of center $c \in M$ and radius $r \in \mathbb{R}_{\geq 0}$ by $\mathrm{B}(c; r)$ and $\mathrm{B}[c; r]$, respectively. (Notice that $\mathrm{B}(c; r)$ is always open and $\mathrm{B}[c; r]$ is always closed and $\mathrm{B}(c; r) \subseteq \mathrm{B}[c; r]$, but there exist metric spaces in which the closure of an open ball $\mathrm{B}(c; r)$ is strictly contained in the closed ball $\mathrm{B}[c; r]$.)

Proposition 1.28. *Every metric space is Hausdorff.*

Definition 1.25. Let $\boldsymbol{M} = (M, |\cdot,\cdot|)$ and $\boldsymbol{M}' = (M, |\cdot,\cdot|')$ be metric spaces. A *Lipschitz* (or *controlled*) transformation from $\boldsymbol{M}$ to $\boldsymbol{M}'$ is a transformation $f : M \to M'$ for which there exists a number $\delta \in \mathbb{R}_{\geq 0}$ such that, for every $p, q \in M$,

$$|f(p), f(q)|' \leq \delta |p, q|.$$

The *distortion* (or *dilation*) of f is the least such number δ, denoted $\langle\!\langle f \rangle\!\rangle$.

Definition 1.26. Let $\boldsymbol{M}$ and $\boldsymbol{M}'$ be metric spaces. A *contraction* from $\boldsymbol{M}$ to $\boldsymbol{M}'$ is a Lipschitz transformation $f : M \to M'$ such that $\langle\!\langle f \rangle\!\rangle < 1$.

Theorem 1.29 (Banach fixed point). *Let $\boldsymbol{M}$ be a non-empty complete metric space and $f : M \to M$ a contraction. There exists a unique fixed point $p \in M$ and, for every point $p' \in M$,*

$$\lim_{n \to \infty} f^n(p') = p.$$

Definition 1.27. Let $\boldsymbol{M} = (M, |\cdot,\cdot|)$ and $\boldsymbol{M}' = (M, |\cdot,\cdot|')$ be metric spaces and $\alpha \in \mathbb{R}_{>0}$. An *$\alpha$-order Hölder* (or *$\alpha$-order controlled*) transformation from $\boldsymbol{M}$ to $\boldsymbol{M}'$ is a transformation $f : M \to M'$ for which there exists a number $\delta \in \mathbb{R}_{\geq 0}$ such that, for every $p, q \in M$,

$$|f(p), f(q)|' \leq \delta |p, q|^{\alpha}.$$

The *α-order distortion* (or *α-order dilation*) of f is the least such number δ, denoted $\langle\!\langle f \rangle\!\rangle_{\alpha}$.

1.2.1 Completeness

Definition 1.28. Let $\boldsymbol{M} = (M, |\cdot,\cdot|)$ be a metric space. A *Cauchy* sequence on $\boldsymbol{M}$ is a sequence $(p_n)_{n \in \mathbb{N}}$ on M that satisfies the following: for every $\varepsilon \in \mathbb{R}_{>0}$, there exists $n_\varepsilon \in \mathbb{N}$ such that, for every $n, n' \in \mathbb{N}$, if $n \geq n_\varepsilon$ and $n' \geq n_\varepsilon$ then $|p_n, p_{n'}| \leq \varepsilon$. A *complete* metric space is a metric space in which every Cauchy sequence has a limit.

Definition 1.29. A *metrizable* topological space is a topological space $\boldsymbol{M} = (M, \mathcal{T})$ for which there exists a metric $|\cdot,\cdot|$ that generates the topology $\mathcal{T}$ of $\boldsymbol{M}$. A *completely metrizable* topological space is a topological space $\boldsymbol{M} = (M, \mathcal{T})$ for which there exists a metric $|\cdot,\cdot|$ that generates the topology $\mathcal{T}$ and $(M, |\cdot,\cdot|)$ is a complete metric space.

Proposition 1.30. *Every closed subspace of a complete metric space is complete.*

Proposition 1.31. *A metrizable space $\boldsymbol{M}$ is $\aleph_0$-generated (second-countable) if, and only if, it is $\aleph_0$-dense.*

Proof. If $\boldsymbol{M}$ is $\aleph_0$-generated, then it is $\aleph_0$-dense (PROPOSITION 1.5). For the converse, let $|\cdot,\cdot|$ be a metric compatible with $\boldsymbol{M}$. If $\boldsymbol{M}$ is $\aleph_0$-dense, let D be a countable dense set and consider the set $\mathcal{B} := \{\mathrm{B}(c;r) \mid p \in D,\ r \in \mathbb{Q}_{>0}\}$. The set $\mathcal{B}$ is a countable basis for $\boldsymbol{M}$. ∎

Proposition 1.32 (Cantor)**.** *Let $\boldsymbol{M}$ be a complete metric space and $M \supseteq F_0 \supseteq F_1 \supseteq \ldots$ a decreasing sequence of non-empty closed sets with diameter $\lim_{n\to\infty} \Theta(F_n) = 0$. Then there exists a point $p_\infty \in M$ such that*

$$\bigcap_{n\in\mathbb{N}} F_n = \{p_\infty\}$$

and, for every sequence $(p_n)_{n\in\mathbb{N}}$ in M such that $p_n \in F_n$,

$$\lim_{n\to\infty} p_n = p_\infty.$$

Proof. For every $n \in \mathbb{N}$, the set F_n is non-empty, so there exists a point $p_n \in F_n$. The sequence $(p_n)_{n\in\mathbb{N}}$ is Cauchy. To see this, take $\varepsilon > 0$. Since $\lim_{n\to\infty} \Theta(F_n) = 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\Theta(F_{n_\varepsilon}) \le \varepsilon$. Now, for every $n, n' \in \mathbb{N}$ such that $n \ge n_\varepsilon$ and $n' \ge n_\varepsilon$, we have $F_n \cup F_{n'} \subseteq F_{n_\varepsilon}$, so

$$|p_n, p_{n'}| \le \Theta(F_{n_\varepsilon}) \le \varepsilon.$$

Since $\boldsymbol{M}$ is complete, there exists a limit point $p_\infty := \lim_{n\to\infty} p_n$. For every $n, n' \in \mathbb{N}$ such that $n' \ge n$, we have $p_{n'} \in F_{n'} \subseteq F_n$; since F_n is closed, this implies that $p_\infty \in F_n$, which in turn implies that $p_\infty \in \bigcap_{n\in\mathbb{N}} F_n$. This shows that $\bigcap_{n\in\mathbb{N}} F_n \neq \mathbb{0}$. Finally, from the fact that

$$\Theta(\bigcap_{n\in\mathbb{N}} F_n) \le \inf_{n\ge 0} \Theta(F_n) = 0$$

it follows that $\bigcap_{n\in\mathbb{N}} F_n = \{p_\infty\}$. ∎

1.2.2 Compactness

Definition 1.30. A *totally bounded* metric space is a metric space $\boldsymbol{M}$ such that, for every radius $r \in \mathbb{R}_{>0}$, there exists a finite open r-cover of M.

Proposition 1.33 (Uniform compactness/Heine-Borel)**.** *A metric space $\boldsymbol{M}$ is compact if, and only if, it is complete and totally bounded.*

Proof. Check [Mun00, Theorem 45.1, p. 276]. ∎

This proposition is true for every uniform space.

Definition 1.31. A *sequentially compact* metric space is a metric space $\boldsymbol{M}$ for which every sequence $(p_n)_{n\in\mathbb{N}}$ in M has a convergent subsequence whose limit is in M.

Proposition 1.34 (Countable uniform compactness)**.** *A metric space $\boldsymbol{M}$ is compact if, and only if, it is sequentially compact.*

Proof. Check [Mun00, Theorem 28.2, p. 179]. ∎

This proposition is true for every first-countable uniform space.

Compact set distance Remember that in a metric space $\boldsymbol{M}$ we denote the set of compact subsets of M by $\mathcal{T}_\blacklozenge$, and the non-empty comapact sets by $\mathcal{T}_{\blacklozenge\supset\emptyset} = \mathcal{T}_\blacklozenge \setminus \{\emptyset\}$, the closed ball of center $c \in M$ and radius $r \in \mathbb{R}_{\geq 0}$ by $\mathrm{B}[c;r]$, also called the closed r-neighborhood of c, and the closed r-neighborhood of a set $C \subseteq M$ by

$$\mathrm{B}[C;r] = \bigcup_{c\in C} \mathrm{B}[c;r].$$

Definition 1.32. Let $\boldsymbol{M}$ be a metric space and $K, K' \subseteq M$ non-empty compact sets. The (*Hausdorff*) *compact set distance* between K and K' is

$$|K,K'|_\blacklozenge := \inf\{r \in \mathbb{R}_{\geq 0} \mid K \subseteq \mathrm{B}[K';r],\ K' \subseteq \mathrm{B}[K;r]\}.$$

The (*Hausdorff*) *compact set distance* on the set $\mathcal{T}_{\blacklozenge\supset\emptyset}$ of non-empty compact sets is the function

$$\begin{aligned} |\cdot,\cdot|_\blacklozenge \colon \mathcal{T}_{\blacklozenge\supset\emptyset} \times \mathcal{T}_{\blacklozenge\supset\emptyset} &\longrightarrow \mathbb{R}_{\geq 0} \\ (K,K') &\longmapsto |K,K'|_\blacklozenge. \end{aligned}$$

The value $|C,C'|_\blacklozenge$ can actually be defined for any non-empty subset of the metric space, but it can infinite, and the function $|\cdot,\cdot|_\blacklozenge$ can fail to be a distance.

Exercise 1.1. Show that

$$|K,K'|_\blacklozenge = \max\{\sup_{p\in K}|p,K'|, \sup_{p'\in K'}|p',K|\}.$$

The function $|\cdot,\cdot|_\blacklozenge$ is in fact a distance over the non-empty closed and bounded subsets of the space, but here we restrict it to the non-empty compact sets because that will be our main use for it.

Proposition 1.35. *Let $\boldsymbol{M}$ be a metric space. The compact set distance $|\cdot,\cdot|_\blacklozenge$ is a distance on $\mathcal{T}_{\blacklozenge\supset\emptyset}$.*

Proof. Notice that, for every $K, K' \in \mathcal{T}_{\blacklozenge\supset\emptyset}$, since K and K' are compact,m they are bounded, so there exists a point $p \in M$ and radius $r \in \mathbb{R}_{>0}$ such that $K \cup K' \subseteq \mathrm{B}(p;r)$, which implis that $|K,K'| < 2r < \infty$.

Now we need to prove separation, symmetry and the triangle inequality.

1. (Separation) Let $K, K' \in \mathcal{T}_{\blacklozenge\supset\emptyset}$. If $K = K'$, then $\mathrm{B}[K;0] = K = K'$ and $\mathrm{B}[K';0] = K' = K$, so $|K,K'|_\blacklozenge = 0$. Conversely, if $|K,K'|_\blacklozenge = 0$, then, for every $r \in \mathbb{R}_{>0}$, $K \subseteq \mathrm{B}[K';r]$ and $K' \subseteq \mathrm{B}[K;r]$. This implies that every point of K is a point of accumulation of K' and vice-versa. Since K and K' are compact in the metric space $\boldsymbol{M}$, they are closed, so $K \subseteq K'$ and $K' \subseteq K$, therefore $K = K'$.
2. (Symmetry) Let $K, K' \in \mathcal{T}_{\blacklozenge\supset\emptyset}$. Then $|K,K'|_\blacklozenge = |K',K|_\blacklozenge$ follows directly from the definition of the compact set distance.
3. (Triangle inequality) Let us denote

$$\begin{aligned} R &:= \{r \in \mathbb{R}_{\geq 0} \mid K \subseteq \mathrm{B}[K'';r],\ K'' \subseteq \mathrm{B}[K;r]\}, \\ R' &:= \{r \in \mathbb{R}_{\geq 0} \mid K \subseteq \mathrm{B}[K';r],\ K' \subseteq \mathrm{B}[K;r]\}, \\ R'' &:= \{r \in \mathbb{R}_{\geq 0} \mid K' \subseteq \mathrm{B}[K'';r],\ K'' \subseteq \mathrm{B}[K';r]\}, \end{aligned}$$

so that $|K,K''|_\blacklozenge = \inf R$, $|K,K'|_\blacklozenge = \inf R'$ and $|K',K''|_\blacklozenge = \inf R''$.

Take $r' \in R'$ and $r'' \in R''$. From the definition of R', we have $K \subseteq \mathrm{B}[K';r']$ and $K' \subseteq \mathrm{B}[K;r']$; from the definition of R'', we have $K' \subseteq \mathrm{B}[K'';r'']$ and $K'' \subseteq \mathrm{B}[K';r'']$.

Take $q \in \mathrm{B}[K';r']$. Then there exists $q' \in K'$ such that $q \in \mathrm{B}[q';r']$, so $|q',q| \leq r'$. Since $K' \subseteq \mathrm{B}[K'';r'']$, there exists $q'' \in K''$ such that $q' \in \mathrm{B}[q'';r'']$, so $|q'',q'| \leq r''$. Then it follows from the triangle inequality for $|\cdot,\cdot|$ that

$$|q'',q| \leq |q'',q'| + |q',q| = r'' + r' = r' + r'',$$

so $q \in \mathrm{B}[q''; r' + r'']$, hence $\mathrm{B}[K'; r'] \subseteq \mathrm{B}[K''; r' + r'']$. Therefore we obtain

$$K \subseteq \mathrm{B}[K'; r'] \subseteq \mathrm{B}[K''; r' + r''].$$

In the same way, we also have $K'' \subseteq \mathrm{B}[K'; r' + r'']$. This shows that $r' + r'' \in R$, so $R' + R'' \subseteq R$, and we conclude that

$$|K, K''|_\blacklozenge = \inf R \leq \inf(R' + R'') = \inf R' + \inf R'' = |K, K'|_\blacklozenge + |K', K''|_\blacklozenge. \qquad \blacksquare$$

Proposition 1.36. *Let **M** be a metric space.*

1. *If **M** is complete, then the metric space $(\mathcal{T}_{\blacklozenge \supset \emptyset}, |\cdot,\cdot|_\blacklozenge)$ is complete.*
2. *If **M** is totally bounded, then the metric space $(\mathcal{T}_{\blacklozenge \supset \emptyset}, |\cdot,\cdot|_\blacklozenge)$ is totally bounded.*
3. *If **M** is compact, then the metric space $(\mathcal{T}_{\blacklozenge \supset \emptyset}, |\cdot,\cdot|_\blacklozenge)$ is compact.*

1.2.3 Iterated function systems

Remember that, for every $k \in \mathbb{N}$, we denote $[\![k]\!] = \{n \in \mathbb{N} \mid n < k\} = \{0, \dots, k-1\}$. For each $n \in \mathbb{N}$, the elements of $[\![k]\!]^n$ are finite sequences denoted $w = (w_0, \dots, w_{n-1})$, and the elements of $w \in [\![k]\!]^{\mathbb{N}}$ are inifite sequences denoted $w = (w_0, w_1, \dots)$, whose restriction to the first n terms is denoted $w|_n = (w_0, \dots, w_{n-1})$.

Definition 1.33. Let $\boldsymbol{M}$ be a metric space. An *iterated function system* (*IFS*) on M is a sequence $f = (f_0, \dots, f_{k-1})$ (with $k \geq 2$) of transformations $c_i : M \to M$. For every sequence $w \in [\![k]\!]^n$, we denote

$$f_w := f_{w_0} \circ \cdots \circ f_{w_{n-1}}.$$

The *direct image of* f is the function

$$\begin{aligned} f : 2^M &\longrightarrow 2^M \\ X &\longmapsto \bigcup_{i=0}^{k-1} f_i(X). \end{aligned}$$

A *contracting iterated function system* is an iterated function system $c = (c_0, \dots, c_{k-1})$ for which every c_i is a contraction.

Definition 1.34. An *attractor* of f is a non-empty compact set $A \subseteq M$ such that

$$A = f(A) = \bigcup_{i=0}^{k-1} f_i(A).$$

The following proposition is based on [Fal14, Theorem 9.1, p. 135].

Proposition 1.37 (Existence and uniqueness of the attractor)**.** *Let **M** be a non-empty compact metric space and $c = (c_0, \dots, c_{k-1})$ a contracting iterated function system on M. There exists a unique attractor $A \subseteq M$ of c and, for every non-empty compact set $K \subseteq M$ that is positive c_i-invariant for every $i \in [\![k]\!]$,*

$$A = \bigcap_{n=0}^{\infty} c^n(K).$$

Proof. Let us first note that the metric space $\boldsymbol{M}$ is complete, because it is compact (PROPOSITION 1.33), so the metric space $(\mathcal{T}_{\bullet\supset\emptyset}, |\cdot,\cdot|_{\bullet})$ is also complete (PROPOSITION 1.36), and it is non-empty because $M \in \mathcal{T}_{\bullet\supset\emptyset}$.

We denote the distortion of c_i by $\langle\langle c_i \rangle\rangle$. Since c_i is a contraction for every $i \in [\![k]\!]$, then $0 \leq \max_{0\leq i\leq k-1}\langle\langle c_i \rangle\rangle < 1$. Since, for every non-empty compact sets $K, K' \in \mathcal{T}_{\bullet\supset\emptyset}$, we have

$$|c(K), c(K)|_{\bullet} = \left|\bigcup_{i=0}^{k-1} c_i(K), \bigcup_{i=0}^{k-1} c_i(K')\right|_{\bullet} \leq \max_{0\leq i<k} |c_i(K), c_i(K')|_{\bullet} \leq \left(\max_{0\leq i<k} \langle\langle c_i \rangle\rangle\right) |K, K'|_{\bullet},$$

this shows that $c\colon \mathcal{T}_{\bullet\supset\emptyset} \to \mathcal{T}_{\bullet\supset\emptyset}$ is a contraction.

An attractor of c is a fixed point of $c\colon \mathcal{T}_{\bullet\supset\emptyset} \to \mathcal{T}_{\bullet\supset\emptyset}$, because it satisfies $c(A) = A$ by definition. THEOREM 1.29 implies that there exists a unique fixed point $A \in \mathcal{T}_{\bullet\supset\emptyset}$, that is, A is the unique attractor of c, and that, for every $K \in \mathcal{T}_{\bullet\supset\emptyset}$, $\lim_{n\to\infty} c^n(K) = A$.

If $K \in \mathcal{T}_{\bullet\supset\emptyset}$ is positive c_i-invariant for every $i \in [\![k]\!]$, then $c(K) = \bigcup_{i=0}^{k-1} c_i(K) \subseteq K$. This implies that $(c^n(K))_{n\in\mathbb{N}}$ is a decreasing sequence, so

$$\bigcap_{n=0}^{\infty} c^n(K) = \lim_{n\to\infty} c^n(K) = A. \qquad \blacksquare$$

Lemma 1.38. *Let $\boldsymbol{M}$ be a complete metric space, $c = (c_0, \dots, c_{k-1})$ a contracting iterated function system on M and $A \subseteq M$ its attractor. If the contractions c_i are injetive and there is a non-empty compact set K that is positive c_i-invariant for every $i \in [\![k]\!]$ and such that the sets $c_i(A)$ are pairwise disjoint, then, for every $n, m \in \mathbb{N}$, $w \in [\![k]\!]^n$ and $w' \in [\![k]\!]^{n+m}$, if $w \neq w'|_n$ then*

$$c_w(K) \cap c_{w'}(K) = \emptyset.$$

Proposition 1.39 (Coding the attractor). *Let $\boldsymbol{M}$ be a complete metric space, $c = (c_0, \dots, c_{k-1})$ a contracting iterated function system on M and $A \subseteq M$ the attractor of c. For every point $x \in A$, there exists a sequence $w \in [\![k]\!]^{\mathbb{N}}$ such that, for every non-empty compact set $K \subseteq M$ that is positive c_i-invariant for every $i \in [\![k]\!]$,*

$$\{x\} = \bigcap_{n=0}^{\infty} c_{w|_n}(K).$$

Besides that,

$$A = \bigcup_{w\in[\![k]\!]^{\mathbb{N}}} \bigcap_{n=0}^{\infty} c_{w|_n}(K).$$

Proof. Notice that

$$c^n(K) = \bigcup_{w\in[\![k]\!]^n} c_w(K).$$

Since $A = \bigcap_{n=0}^{\infty} c^n(K)$ by PROPOSITION 1.37, then

$$A = \bigcap_{n=0}^{\infty} \bigcup_{w\in[\![k]\!]^n} c_w(K).$$

Let $x \in A$. Then, for every $n \in \mathbb{N}$, there exists $w^{(n)} \in [\![k]\!]^n$ such that $x \in c_{w^{(n)}}(K)$.

$\blacksquare$

Notice that

$$\Theta(c_{w|_n}(K)) = \Theta(c_{w_0} \circ \cdots \circ c_{w_{n-1}}(K)) \leq \langle\!\langle c_{w_0} \rangle\!\rangle \cdots \langle\!\langle c_{w_{n-1}} \rangle\!\rangle \, \Theta(K) \leq \left(\max_{0 \leq i < k} \langle\!\langle c_i \rangle\!\rangle \right)^n \Theta(K),$$

so $0 \leq \max_{0 \leq i < k} \langle\!\langle c_i \rangle\!\rangle < 1$ implies that $\lim_{n \to \infty} \Theta(c_{w|_n}(K)) = 0$. Since K is positive c_i-invariant for every $i \in [\![k]\!]$, then, for every $n \in \mathbb{N}$, we have $c_{w|_{n+1}}(K) \subseteq c_{w_n}(K)$. Since the contractions c_i are continuous and K is compact, then $c_i(K)$ is compact, hence closed. So PROPOSITION 1.32 implies that $\bigcap_{n=0}^{\infty} c_{w|_n}(K)$ consists of a single point, which we denote p_w.

Proposition 1.40. *Let $\boldsymbol{M}$ be a complete metric space, $c = (c_0, \ldots, c_{k-1})$ a contracting iterated function system on M and $A \subseteq M$ the attractor of c. If every c_i is injective and the sets $c_i(A)$ are pairwise disjoint, then the attractor A is totally disconnected.*

The Cantor set Let $\mathbb{I} \coloneqq [0, 1]$ denote the unit interval in the real line $\mathbb{R}$ and, for each $k \in \mathbb{N}_{\geq 1}$, $[\![k]\!] \coloneqq \{n \in \mathbb{N} \mid n < k\} = \{0, \ldots, k-1\}$. For each $w \in [\![k]\!]$, define the transformation

$$\begin{aligned} c_w \colon \mathbb{I} &\longrightarrow \mathbb{I} \\ x &\longmapsto \frac{2w + x}{2k - 1}. \end{aligned}$$

The transformations c_0 and c_1 are contractions with factor $\frac{1}{2k-1}$, and they are injective.

We define, for each finite sequence (of length $n \in \mathbb{N}$) $w = (w_0, \ldots, w_{n-1}) \in [\![k]\!]^n$, the contraction

$$c_w \coloneqq c_{w_0} \circ \cdots \circ c_{w_{n-1}}.$$

The empty sequence just gives the identity $c_{\mathbb{0}} = \mathrm{I}$ and the unit interval $c_{\mathbb{0}}(\mathbb{I}) = \mathbb{I}$, and the first iterates gives

$$c_w(\mathbb{I}) = \left[\frac{2w}{2k - 1}, \frac{2w + 1}{2k - 1} \right].$$

Since c_0 and c_1 are contractions with factor $\frac{1}{2k-1}$, we have $\Theta(c_w(\mathbb{I})) = \frac{1}{(2k-1)^n}$. In general, for $w \in [\![k]\!]^n$, we have

$$c_w(\mathbb{I}) = \left[\bigplus_{i=0}^{n-1} \frac{2w_i}{(2k-1)^{i+1}}, \bigplus_{i=0}^{n-1} \frac{2w_i}{(2k-1)^{i+1}} + \frac{1}{(2k-1)^n} \right].$$

Lemma 1.41. *Let $n, m \in \mathbb{N}$, $w \in [\![k]\!]^n$ and $w' \in [\![k]\!]^{n+m}$. If $w \neq w'|_n$ then*

$$c_w(\mathbb{I}) \cap c_{w'}(\mathbb{I}) = \mathbb{0}.$$

Proof. We first prove the case $m = 0$ by induction on n. The initial case $n = 0$ is vacuosly true. For the sucessor case, suppose this is true for every natural numbers from 0 to n and take $w, w' \in [\![k]\!]^{n+1}$ such that $w \neq w'$. We consider 2 cases. (1) If $w_0 \neq w'_0$, then $c_{w_0}(\mathbb{I}) \cap c_{w'_0}(\mathbb{I}) = \mathbb{0}$. Since $c_w(\mathbb{I}) = c_{w_0}(c_{(w_1,\ldots,w_n)}(\mathbb{I})) \subseteq c_{w_0}(\mathbb{I})$ and $c_{w'}(\mathbb{I}) = c_{w'_0}(c_{(w'_1,\ldots,w'_n)}(\mathbb{I})) \subseteq c_{w'_0}(\mathbb{I})$, it follows that $c_w(\mathbb{I}) \cap c_{w'}(\mathbb{I}) = \mathbb{0}$. (2) If $w_0 = w'_0$, then we must have $(w_1, \ldots, w_n) \neq (w'_1, \ldots, w'_n)$, so, by the induction hypothesis, we obtain $c_{(w_1,\ldots,w_n)}(\mathbb{I}) \cap c_{(w'_1,\ldots,w'_n)}(\mathbb{I}) = \mathbb{0}$. Since $c_{w_0} = c_{w'_0}$ is injetive, then

$$c_w(\mathbb{I}) \cap c_{w'}(\mathbb{I}) = c_{w_0}(c_{(w_1,\ldots,w_n)}(\mathbb{I})) \cap c_{w'_0}(c_{(w'_1,\ldots,w'_n)}(\mathbb{I})) = c_{w_0}(c_{(w_1,\ldots,w_n)}(\mathbb{I}) \cap c_{(w'_1,\ldots,w'_n)}(\mathbb{I})) = c_{w_0}(\mathbb{0}) = \mathbb{0}.$$

This completes the induction on n for $m = 0$. We now prove the result by induction on m. The initial case $m = 0$ has just been proved. For the successor case, suppose that it is true for every natural number from 0 to m and take $w \in [\![k]\!]^n$ and $w' \in [\![k]\!]^{n+m+1}$. Notice that $c_{w'}(\mathbb{I}) \subseteq c_{(w'_0,\ldots,w'_{n-1})}(\mathbb{I})$, so if $w \neq (w'_0, \ldots, w'_{n-1})$, then from the case $m = 0$ it follows that $c_w(\mathbb{I}) \cap c_{(w'_0,\ldots,w'_{n-1})}(\mathbb{I}) = \mathbb{0}$, so $c_w(\mathbb{I}) \cap c_{w'}(\mathbb{I}) = \mathbb{0}$. ■

We also define, for each sequence $w \in [\![k]\!]^{\mathbb{N}}$, the set

$$c_w(\mathbb{I}) := \bigcap_{n\in\mathbb{N}} c_{(w_0,\dots,w_{n-1})}(\mathbb{I}) = \bigcap_{n\in\mathbb{N}} c_{w|_n}(\mathbb{I}).$$

We have a decreasing sequence of closed sets

$$\mathbb{I} \supseteq c_{w_0}(\mathbb{I}) \supseteq \dots \supseteq c_{w|_n}(\mathbb{I}) \supseteq \dots$$

with diameter $\Theta(c_{(w_0,\dots,w_{n-1})}(\mathbb{I})) = \frac{1}{(2k-1)^n} \to 0$, so PROPOSITION 1.32 guarantees that there exists a unique point $p_w \in \mathbb{I}$ such that $c_w(\mathbb{I}) = \{p_w\}$.

Lemma 1.42. *Let $w, w' \in [\![k]\!]^{\mathbb{N}}$. If $w \neq w'$, then $c_w(\mathbb{I}) \cap c_{w'}(\mathbb{I}) = \emptyset$.*

Proof. There exists $n \in \mathbb{N}$ such that $w_n \neq w'_n$, so $c_{(w_0,\dots,w_n)}(\mathbb{I}) \cap c_{(w'_0,\dots,w'_n)}(\mathbb{I}) = \emptyset$ (LEMMA 1.41); since $c_w(\mathbb{I}) \subseteq c_{(w_0,\dots,w_n)}(\mathbb{I})$ and $c_{w'}(\mathbb{I}) \subseteq c_{(w'_0,\dots,w'_n)}(\mathbb{I})$, it follows that $c_w(\mathbb{I}) \cap c_{w'}(\mathbb{I}) = \emptyset$. ■

Finally, for each $n \in \mathbb{N}$ we define the set

$$C^{(n)} := \bigcup_{w\in[\![k]\!]^n} c_w(\mathbb{I}).$$

The set $C^{(n)}$ is a disjoint union of the sets $c_w(\mathbb{I})$, since these sets are pairwise disjoint.

Definition 1.35. The *Cantor set* is the set

$$\mathbb{K}_k := \bigcap_{n\in\mathbb{N}} C^{(n)}.$$

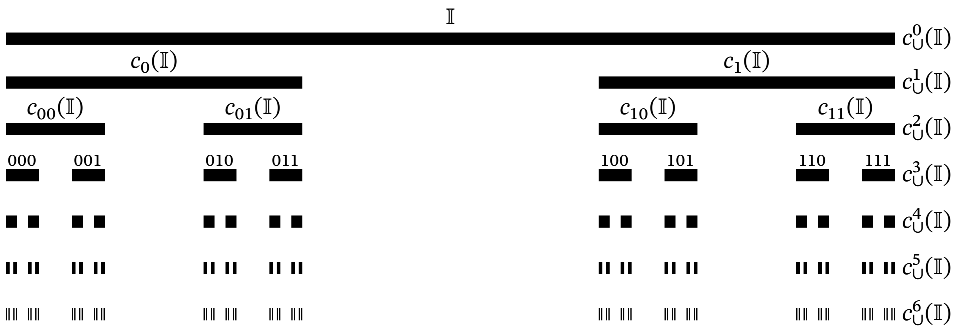

Figure 1. The first seven pre-fractals of the Cantor set $\mathbb{K}_2$.

Proposition 1.43. *The transformation*

$$\begin{aligned} h : [\![k]\!]^{\mathbb{N}} &\longrightarrow \mathbb{K}_k \\ w &\longmapsto \sum_{i=0}^{\infty} \frac{2w_i}{(2k-1)^{i+1}} \end{aligned}$$

is a homeomorphism between $[\![k]\!]^{\mathbb{N}}$ (with the product topology) and $\mathbb{K}_k$ (with the subspace topology inherited from $\mathbb{I}$).

Proof. We need to show that h is well-defined in the sense that its image is contained in $\mathbb{K}_k$, that h is injective and surjective, and that h is continuous and its inverse is continuous.

1. ($h([\![k]\!]^{\mathbb{N}}) \subseteq \mathbb{K}_k$) For every $n \in \mathbb{N}$ and $w \in [\![k]\!]^n$, define the point $p_w := \bigplus_{i=0}^{n-1} \frac{2w_i}{(2k-1)^{i+1}}$. We will prove by induction on n that $p_w \in c_w(\mathbb{I})$. For the initial case, $p_0 = 0 \in \mathbb{I} = c_{\emptyset}(\mathbb{I})$. For the successor case, assume it is true for every natural number from 0 to n, and take $w \in [\![k]\!]^{n+1}$. By the induction hypothesis, $p_{(w_1,\dots,w_n)} \in c_{(w_1,\dots,w_n)}(\mathbb{I})$. Calculating c_{w_0} at this point, we obtain

$$\begin{aligned} c_{w_0}(p_{(w_1,\dots,w_n)}) &= c_{w_0}\left(\bigplus_{i=0}^{n-1} \frac{2w_i}{(2k-1)^{i+1}}\right) \\ &= \frac{2w_0}{(2k-1)} + \frac{1}{(2k-1)} \bigplus_{i=0}^{n-1} \frac{2w_i}{(2k-1)^{i+1}} \\ &= \frac{2w_0}{(2k-1)} + \bigplus_{i=1}^{n} \frac{2w_i}{(2k-1)^{i+1}} \\ &= \bigplus_{i=0}^{n} \frac{2w_i}{(2k-1)^{i+1}} = p_w, \end{aligned}$$

so $p_w \in c_{w_0}(c_{(w_1,\dots,w_n)}(\mathbb{I})) = c_w(\mathbb{I})$. This completes the induction proof.
   Now take $w \in [\![k]\!]^{\mathbb{N}}$. For every $n \in \mathbb{N}$, we have $p_{(w_0,\dots,w_{n-1})} \in c_{(w_0,\dots,w_{n-1})}(\mathbb{I})$ so it follows from PROPOSITION 1.32 that $\lim_{n\to\infty} p_{(w_0,\dots,w_{n-1})} \in c_w(\mathbb{I})$, hence we conclude that

$$h(w) = \bigplus_{i=0}^{\infty} \frac{2w_i}{(2k-1)^{i+1}} = \lim_{n\to\infty} \bigplus_{i=0}^{n-1} \frac{2w_i}{(2k-1)^{i+1}} = \lim_{n\to\infty} p_{(w_0,\dots,w_{n-1})} = p_w \in \mathbb{K}_k.$$

2. (h is injective) Take $w, w' \in [\![k]\!]^{\mathbb{N}}$ such that $w \neq w'$. We have just showed that $h(w) = p_w \in c_w(\mathbb{I})$ and $h(w') = p_{w'} \in c_{w'}(\mathbb{I})$. Since $c_w(\mathbb{I}) \cap c_{w'}(\mathbb{I}) = \emptyset$, it follows that $h(w) \neq h(w')$.
3. (h is surjective) Take $p \in \mathbb{K}_k$. We will find a sequence $w \in [\![k]\!]^{\mathbb{N}}$ such that $h(w) = p$. By the definition of $\mathbb{K}_k$, for every $n \in \mathbb{N}$, $p \in C^{(n)}$ and, since $C^{(n)}$ is a disjoint union of the sets $c_w(\mathbb{I})$ with $w \in [\![k]\!]^n$, there exists a unique $w^{(n)} \in [\![k]\!]^n$ such that $p \in c_{w^{(n)}}(\mathbb{I})$, so

$$p \in \bigcap_{n\in\mathbb{N}} c_{w^{(n)}}(\mathbb{I}).$$

   This implies that, for every $n \in \mathbb{N}$, $c_{w^{(n)}}(\mathbb{I}) \cap c_{w^{(n+1)}}(\mathbb{I}) \neq \emptyset$, so it follows from LEMMA 1.41 that the first n entries of $w^{(n+1)}$ coincide with $w^{(n)}$. Then there exists a sequence $w \in [\![k]\!]^{\mathbb{N}}$ such that, for every $n \in \mathbb{N}$, $w^{(n)} = (w_0, \dots, w_{n-1})$, therefore

$$p \in \bigcap_{n\in\mathbb{N}} c_{w^{(n)}}(\mathbb{I}) = \bigcap_{n\in\mathbb{N}} c_{(w_0,\dots,w_{n-1})}(\mathbb{I}) = c_w(\mathbb{I}) = \{p_w\}$$

   and we conclude that $p = p_w = h(w)$.
4. (h is continuos) It is sufficient to show that, given a subbasis of $\mathbb{K}_k$, the inverse image of each subbasic set by h is open in $[\![k]\!]^{\mathbb{N}}$. Notice that, for every $n \in \mathbb{N}$, since the sets $c_w(\mathbb{I})$ with $w \in [\![k]\!]^n$ are pairwise disjoint, we can find pairwise disjoint open neighborhoods $A_w \subseteq \mathbb{I}$ of $c_w(\mathbb{I})$, and also pairwise disjoint open neighborhoods $F_w \subseteq \mathbb{I}$ of $c_w(\mathbb{I})$ (this is so because $\mathbb{I}$ is a normal space). Since $c_w(\mathbb{I}) \cap A_w = c_w(\mathbb{I}) = c_w(\mathbb{I}) \cap F_w$, this shows that $c_w(\mathbb{I})$ is both an open and a closed set in the subspace topology of $\mathbb{K}_k$. This shows that $\bigcup n \in \mathbb{N}\{c_w(\mathbb{I}) \mid w \in [\![k]\!]^n\}$ is a subbasis of $\mathbb{K}_k$. A subbasis of $[\![k]\!]^{\mathbb{N}}$ is given by the cylinders $\mathrm{C}_{0,\dots,n-1}[w_0, \dots, w_{n-1}]$.

We will show that, for every $w \in [\![k]\!]^n$,

$$h^{-1}(c_w(\mathbb{I})) = \mathrm{C}_{0,\dots,n-1}[w_0,\dots,w_{n-1}] = \bigcap_{i=0}^{n-1}\{w' \in [\![k]\!]^{\mathbb{N}} \mid w'_i = w_i\}.$$

Since $c_w(\mathbb{I}) = [\dotplus_{i=0}^{n-1}\frac{2w_i}{(2k-1)^{i+1}}, \dotplus_{i=0}^{n-1}\frac{2w_i}{(2k-1)^{i+1}} + \frac{1}{(2k-1)^n}]$, we have that, for every $w' \in [\![k]\!]^{\mathbb{N}}$, $h(w') \in c_w(\mathbb{I})$ if, and only if,

$$\dotplus_{i=0}^{n-1}\frac{2w_i}{(2k-1)^{i+1}} \le \dotplus_{i=0}^{\infty}\frac{2w'_i}{(2k-1)^{i+1}} \le \dotplus_{i=0}^{n-1}\frac{2w_i}{(2k-1)^{i+1}} + \frac{1}{(2k-1)^n},$$

which is equivalent to

$$(1) \qquad \dotplus_{i=0}^{n-1}\frac{2(w_i - w'_i)}{(2k-1)^{i+1}} \le \dotplus_{i=n}^{\infty}\frac{2w'_i}{(2k-1)^{i+1}} \le \dotplus_{i=0}^{n-1}\frac{2(w_i - w'_i)}{(2k-1)^{i+1}} + \frac{1}{(2k-1)^n}.$$

If $w' \in \mathrm{C}_{0,\dots,n-1}[w_0,\dots,w_{n-1}]$, then for every $i \in \{0,\dots,n-1\}$ we have $w'_i = w_i$, so $\dotplus_{i=0}^{n-1}\frac{2(w_i-w'_i)}{(2k-1)^{i+1}} = 0$. Since

$$0 \le \dotplus_{i=n}^{\infty}\frac{2w'_i}{(2k-1)^{i+1}} \le \dotplus_{i=n}^{\infty}\frac{2}{(2k-1)^{i+1}} = \frac{1}{(2k-1)^n},$$

it follows from FORMULA (1) that $h(w') \in c_w(\mathbb{I})$, so $w' \in h^{-1}(c_w(\mathbb{I}))$.

If $w' \in h^{-1}(c_w(\mathbb{I}))$, then $h(w') \in c_w(\mathbb{I})$, so FORMULA (1) holds. If $(w'_0,\dots,w'_{n-1}) \neq (w_0,\dots,w_{n-1})$, there is some $k \in \{0,\dots,n-1\}$ such that $w'_k \neq w_k$, and we have a contradiction. Since we have shown the inverse image of every subbasic set is a cylinder in $[\![k]\!]^{\mathbb{N}}$, we conclude that h is continuous.

5. (h^{-1} is continuous) The space $[\![k]\!]^{\mathbb{N}}$ is compact, because it is a product of compact spaces (PROPOSITION 1.23), and $\mathbb{K}_k$ is Hausdorff, because $\mathbb{I}$ is Hausdorff (PROPOSITION 1.3), so it follows from PROPOSITION 1.20 that h^{-1} is also continuous.

■

The construction of the Cantor set $\mathbb{K}_k$ by this iterated function system can be used to show that its Hausdorff dimension is $\log(k)/\log(2k-1)$ and its measure is stricty positive and finite [Fal14, Theorem 9.3, p. 140]. The limit when $k \to 1$ does not give the correct dimension of $\mathbb{K}_1 = \mathbb{I}$, though. Besides that, we can modify the construction of $\mathbb{K}_2$ by taking any real number $r \in]0, 1/2]$ and defining, for each $w \in [\![2]\!]$, the contraction

$$\begin{aligned} c_w : \mathbb{I} &\longrightarrow \mathbb{I} \\ x &\longmapsto r((r^{-1}-1)w + x), \end{aligned}$$

which has contraction constant r. This iterated function system will generate a Cantor set of Hausdorff dimension $-\log(2)/\log(r)$, which attains every value in $]0, 1]$ as r varies in $]0, 1/2]$.

1.2.4 Examples of distances relevant for dynamics

Distance in countable product of metric spaces For the next definition, we adopt the notation $\alpha^{\infty} = 0$ for any $\alpha \in]0, 1[$. Also, remember that $\inf \emptyset = \infty$.

Definition 1.36. Let $(\boldsymbol{M}_i)_{i\in\mathbb{N}} = (M_i, |\cdot,\cdot|_i)_{i\in\mathbb{N}}$ be a countable collection of metric spaces with diameter equal to 1, $\boldsymbol{M} \coloneqq \bigtimes_{i\in\mathbb{N}} \boldsymbol{M}_i$ and $\alpha \in \,]0,1[$. The *ultrametric* (with weight α) in M is the function

$$\begin{aligned} |\cdot,\cdot|_\wedge \colon M \times M &\longrightarrow \mathbb{R}_{\geq 0} \\ (p,p') &\longmapsto |p,p'|_\wedge \coloneqq \alpha^{\inf\{i\in\mathbb{N} \mid |p_i,p'_i|_i \neq 0\}}. \end{aligned}$$

Proposition 1.44. *Let $(\boldsymbol{M}_i)_{i\in\mathbb{N}} = (M_i, |\cdot,\cdot|_i)_{i\in\mathbb{N}}$ be a countable collection of metric spaces (with diameter equal to 1), $\boldsymbol{M} \coloneqq \bigtimes_{i\in\mathbb{N}} \boldsymbol{M}_i$ and $\alpha \in \,]0,1[$. The ultrametric (with weight α) $|\cdot,\cdot|_\wedge$ in M is an ultrametric in Mthat generates the product topology on $\boldsymbol{M}$.*

Proof. We start by proving that $|\cdot,\cdot|_\wedge$ is well-defined in the sense that the codomain of $|\cdot,\cdot|_\wedge$ is $\mathbb{R}_{\geq 0}$. Let $p, p' \in M$. Since $0 < \alpha < 1$ and $\infty \geq \inf\{i \in \mathbb{N} \mid |p_i, p'_i|_i \neq 0\} \geq 0$, it follows that

$$0 = \alpha^\infty \leq \alpha^{\inf\{i\in\mathbb{N} \mid |p_i,p'_i|_i \neq 0\}} \leq \alpha^0 = 1,$$

so $0 \leq |p,p'|_\wedge \leq 1$.

We now show that it is an ultrametric:

1. (Separation) Let $p, p' \in M$. Notice that $|p,p'|_\wedge = \alpha^{\inf\{i\in\mathbb{N} \mid |p_i,p'_i|_i \neq 0\}} = 0$ if, and only if, $\inf\{i \in \mathbb{N} \mid |p_i, p'_i|_i \neq 0\} = \infty$, which holds if, and only if, $\{i \in \mathbb{N} \mid |p_i, p'_i|_i \neq 0\} = \emptyset$. Since, for every $i \in \mathbb{N}$, the distance $|\cdot,\cdot|_i$ satisfies separation, then $|p_i, p'_i| = 0$ holds if, and only if, $p_i = p'_i$, which implies that $\{i \in \mathbb{N} \mid |p_i, p'_i|_i \neq 0\} = \emptyset$ is equivalent to $p = p'$.
2. (Symmetry) Let $p, p' \in M$. For every $i \in \mathbb{N}$, the distance $|\cdot,\cdot|_i$ is symmetric, so it follows that

   $$\{i \in \mathbb{N} \mid |p_i, p'_i|_i \neq 0\} = \{i \in \mathbb{N} \mid |p'_i, p_i|_i \neq 0\},$$

   which implies that $|p,p'|_\wedge = |p',p|_\wedge$.
3. (Ultrametric inequality) For every $p, p' \in M$, denote

   $$\iota(p,p') \coloneqq \inf\{i \in \mathbb{N} \mid |p_i, p'_i|_i \neq 0\}.$$

   Let $p, p', p'' \in M$. We first show a relation between the values $\iota(p,p'')$, $\iota(p,p')$ and $\iota(p',p'')$. For every $i \in \mathbb{N}$ such that $i < \min\{\iota(p,p'), \iota(p',p'')\}$, it holds that $|p_i, p'_i|_i = 0$ and $|p'_i, p''_i|_i = 0$, so it follows from the triangular inequality for $|\cdot,\cdot|_i$ that $|p_i, p''_i| \leq |p_i, p'_i|_i + |p'_i, p''_i|_i = 0$. This shows that

   $$\{i \in \mathbb{N} \mid |p_i, p''_i|_i \neq 0\} \subseteq \{i \in \mathbb{N} \mid |p_i, p'_i|_i \neq 0\} \cup \{i \in \mathbb{N} \mid |p'_i, p''_i|_i \neq 0\},$$

   which implies that

   $$\iota(p,p'') \geq \min\{\iota(p,p'), \iota(p',p'')\}.$$

   Since $\alpha \in \,]0,1[$, from this relation we conclude that

   $$\begin{aligned} |p,p''|_\wedge &= \alpha^{\iota(p,p'')} \\ &\leq \alpha^{\min\{\iota(p,p'),\iota(p',p'')\}} \\ &= \max\{\alpha^{\iota(p,p')}, \alpha^{\iota(p',p'')}\} \\ &= \max\{|p,p'|_\wedge, |p',p''|_\wedge\}. \end{aligned}$$

Finally, we show that it generates the product topology. ■

Two important examples fit in the framework of PROPOSITION 1.44. When we take an integer $n \in \mathbb{Z}_{>0}$ and $M_i = \{0, \dots, n-1\}$, we have the product space $\Sigma_n = \{0, \dots, n-1\}^{\mathbb{Z}_{\geq 0}}$, the usual *unilateral symbolic space* in n symbols, the standard model space in symbolic dynamics. Taking $\alpha = 2^{-1}$ gives a commonly used distance on Σ_n. The other is a relation to p-adic numbers which we will not expose here.

Definition 1.37. Let $(\boldsymbol{M}_i)_{i\in\mathbb{N}} = (M_i, |\cdot,\cdot|_i)_{i\in\mathbb{N}}$ be a countable collection of metric spaces with diameter equal to 1, $M := \bigtimes_{i\in\mathbb{N}} M_i$ and $\alpha \in \,]0,1[$. The *distance* (with weight α)

Distance in commutative groups and quotients

Definition 1.38. Let $\boldsymbol{G} = (G, +, -, 0)$ be a commutative group. A *translation invariant* distance on $\boldsymbol{G}$ is a distance $|\cdot,\cdot|$ on $\boldsymbol{G}$ such that, for every $g, g', h \in G$,

$$|g + h, g' + h'| = |g, g'|.$$

Definition 1.39. Let $\boldsymbol{G} = (G, +, -, 0)$ be a commutative group, $|\cdot,\cdot|\colon G \times G \to \mathbb{R}_{\geq 0}$ a translation invariant metric on $\boldsymbol{G}$, and $\boldsymbol{H} \trianglelefteq \boldsymbol{G}$ a closed (normal) subgroup. The *quotient distance* on the quotient group $\boldsymbol{G}/\boldsymbol{H}$ is the function

$$\begin{aligned} |\cdot,\cdot|_{G/H}\colon\ & G/H \times G/H \longrightarrow \mathbb{R}_{\geq 0} \\ & (g+H, g'+H) \longmapsto |g+H, g'+H|_{G/H} := \inf_{h,h'\in H} |g+h, g'+h'|. \end{aligned}$$

Proposition 1.45. *Let $\boldsymbol{G} = (G, +, -, 0)$ be a commutative group, $|\cdot,\cdot|$ a (left) translation invariant metric on $\boldsymbol{G}$, and $\boldsymbol{H} \trianglelefteq \boldsymbol{G}$ a closed (normal) subgroup. The (right) quotient distance $|\cdot,\cdot|_{G/H}$ is a (right) translation invariant distance on $\boldsymbol{G}/\boldsymbol{H}$.*

Proof. We first prove that $|\cdot,\cdot|_{G/H}$ is well-defined on the equivalence classes of G/H. Let $g+H, g'+H \in G/H$. Take elements $g + k \in g + H$ and $g' + k' \in g' + H$, with $k, k' \in H$. For any $h, h' \in H$, we have $k + h', k' + h' \in H$, so

$$\inf_{h,h'\in H} |g + h, g' + h'| = \inf_{h,h'\in H} |g + k + h, g' + k' + h'|.$$

We now prove each property of a distance.

1. (Separation) Let $g + H, g' + H \in G/H$. If $|g + H, g' + H|_{G/H} = 0$, then there exists a sequence $(h_n, h'_n)_{n\in\mathbb{N}}$ in H^2 such that $\lim_{n\to\infty} |g + h_n, g' + h'_n| = 0$. This implies that $\lim_{n\to\infty}(g + h_n) = \lim_{n\to\infty}(g' + h'_n)$, so $g' - g = \lim_{n\to\infty}(h_n - h'_n)$. Since H is closed, it follows that $(g' - g) \in H$, so $g + H = g' + H$. If $g + H = g' + H$, then it follows from the (left) translation invariance of $|\cdot,\cdot|$ that

$$|g + H, g' + H|_{G/H} = |g + H, g + H|_{G/H} = \inf_{h,h'\in H} |g + h, g + h'| = \inf_{h,h'\in H} |h, h'| = 0.$$

2. (Symmetry) Follows trivially from the symmetry of $|\cdot,\cdot|$.
3. (Triangle inequality) Follows trivially from the triangle inequality of $|\cdot,\cdot|$.

Finally, we prove that $|\cdot,\cdot|_{G/H}$ is (right) translation invariant. Let $g + H, g' + H, k + H \in G/H$. Then $(g + H) + (k + H) = g + k + H$, $(g' + H) + (k + H) = g' + k + H$ and

$$|g+k+H, g'+k+H|_{G/H} = \inf_{h,h'\in H} |g+k+h, g'+k+h'| = \inf_{h,h'\in H} |g+h, g'+h'| = |g+H, g'+H|_{G/H},$$

so $|(g + H) + (k + H), (g' + H) + (k + H)|_{G/H} = |g + H, g' + H|_{G/H}$. ■

1.3 Measure spaces

We denote a measure space by $\boldsymbol{X} = (X, \mathcal{M}, m)$, being X the set of points of the space, $\mathcal{M}$ its σ-algebra and $m : \mathcal{M} \to \mathring{\mathbb{R}}_{\geq 0}$ its measure. We use in $\mathbb{R}^d$ the standard Lebesgue σ-algebra $\mathcal{M}_{\mathbb{R}^d}$ and measure v^d.

Proposition 1.46 (Drawer principle for measure spaces)**.** *Let $\boldsymbol{X}$ a measure space and $(M_i)_{i\in\mathbb{N}}$ a countable sequence of measurable subsets of X. If $\bigcup_{i\in\mathbb{N}} M_i$ has positive measure, then some M_i has positive measure.*

Theorem 1.47 (Lebesgue differentiation theorem)**.** *Let $M \subseteq \mathbb{R}^d$ be a measurable set and $f \in \mathcal{I}^1(M, \mathbb{R})$ an integrable function. For almost every $x \in M$,*

$$f(x) = \lim_{r\to 0} \frac{1}{\mathrm{v}^d(\mathrm{B}(x;r))} \int_{\mathrm{B}(x;r)} f\,\mathrm{v}^d.$$

Proposition 1.48. *Let $M \subseteq \mathbb{R}^d$ be a measurable set. Then M has positive measure if, and only if, for any $\varepsilon > 0$, there exists a ball $B \subseteq \mathbb{R}^d$ such that $m(M \cap B) \geq (1-\varepsilon)\, m(B)$.*

Equivalently, $M \doteq \mathbb{0}$ (is a null set) if, and only if, there exists an $\varepsilon > 0$ such that, for every ball $B \subseteq \mathbb{R}^d$, $m(M \cap B) < (1-\varepsilon)\, m(B)$.

Proof. This is a consequence of the Lebesgue differentiation theorem (THEOREM 1.47). ■

1.3.1 Almost equality

Denote $M \doteq M'$ for $m((M \setminus M') \cup (M' \setminus M)))$.

1.4 Dynamics

Let X be a set and $\boldsymbol{T} = (T, +, 0)$ a monoid. (We denote the set of transformations from X to X by $\mathfrak{S}(X,X)$.) A *monoid action* is a transformation $f : T \to \mathfrak{S}(X,X)$ which satisfies

1. $f^0 = \mathrm{I}$;
2. For every $t, t' \in T$, $f^{t'} \circ f^t = f^{t+t'}$.

A *group action* is a monoid action by a group $\boldsymbol{T} = (T, +, -, 0)$. In particular, for every $t \in T$, the transformation $f^t : X \to X$ of a group action is invertible, since

$$f^t \circ f^{-t} = f^0 = \mathrm{I} = f^0 = f^{-t} \circ f^t.$$

When $\boldsymbol{T}$ is a totally orderd by $\leq$, the *forward cone* of $\boldsymbol{T}$ is the set

$$T_{\geq 0} \coloneqq \{t \in T \mid t \geq 0\},$$

and the *backward cone* of $\boldsymbol{T}$ is the set

$$T_{\leq 0} \coloneqq \{t \in T \mid t \leq 0\}.$$

Definition 1.40. Let X be a set and $\boldsymbol{T} = (T, +, 0, \leq)$ be a totally ordered commutative monoid. A *$\boldsymbol{T}$-time dynamics* on X is a monoid action of $\boldsymbol{T}$ on X:

$$\begin{aligned} f : T &\longrightarrow \mathfrak{S}(X,X) \\ t &\longmapsto f^t : X \longrightarrow X \\ &\qquad\qquad x \longmapsto f^t(x), \end{aligned}$$

The set X is called the *space*, the monoid $\boldsymbol{T}$ the *time*, and the action f the *dynamics* of the *system* $(X, \boldsymbol{T}, f)$.

Let $\boldsymbol{T} = (T, +, -, 0, \leq)$ be a totally ordered commutative group. A $\boldsymbol{T}$*-time invertible dynamics* on X is a group action $f: T \to \mathfrak{S}(X, X)$ on X.

A *discrete-time (invertible) dynamics* is a $\mathbb{Z}_{\geq 0}$-time ($\mathbb{Z}$-time) dynamics f; in this case we denote the dynamics by the time 1 transformation $f: X \to X$ and just take compositions for the action. A *continuous-time (invertible) dynamics* is an $\mathbb{R}_{\geq 0}$-time ($\mathbb{R}$-time) dynamics f; in this case f is a semiflow (flow). (Both $\mathbb{Z}$ and $\mathbb{R}$ are considered with the usual order.)

We mainly consider discrete-time dynamics, but also some continuous-time dynamics occasionally. Other monoids will not be consider for the time of the system, they have been presented here only to show how discrete-time and continuous-time dynamics can be given with a single formulation.

1.4.1 Orbits and invariant sets

Definition 1.41. Let X be a set, f a $\boldsymbol{T}$-time dynamics on X and $x \in X$ a point. The *forward orbit* of x is the set

$$f^{T_{\geq 0}}(x) = \{f^t(x) \mid t \in T_{\geq 0}\}.$$

If f is invertible, the *orbit of x* is the set

$$f^T(x) = \{f^t(x) \mid t \in T\}.$$

Definition 1.42. Let X be a set, f a $\boldsymbol{T}$-time dynamics on X

1. A *forward invariant* set is a set $S \subseteq X$ such that, for all $t \in T_{\geq 0}$, $f^t(S) \subseteq S$;
2. A *backward invariant* set is a set $S \subseteq X$ such that, for all $t \in T_{\leq 0}$, $f^t(S) \subseteq S$;
3. An *invariant* set is a set $S \subseteq X$ such that, for all $t \in T$, $f^t(S) \subseteq S$.

1.5 Examples

1.5.1 Symbolic shift

1.5.2 Odometer

Let us consider the space $\{0, \dots, m-1\}^{\mathbb{N}}$.

2 Baire theorem

2.1 Topologically negligible sets

In the following sections, we will talk about open and closed sets, codense and dense sets, rare and corare sets, meager and comeager sets, and ideals and filters. These pairs are related by duality through the operation of set complementation. Whenever we prove something for one of the member of such pairs, we will omit the proof for the other because it follows directly from the duality principle.

2.1.1 Ideals and filters

Ideals formalize the notion of a negligible element. They are part of the theory of partially ordered sets.

Definition 2.1. Let P be a set. A *partial order* on P is a relation on P that satisfies

1. (Reflexivity) For every $p \in P$, $p \leq p$.
2. (Antisymmetry) For every $p, p' \in P$, if $p \leq p'$ and $p' \leq p$, then $p = p'$.
3. (Transitivity) For every $p, p', p'' \in P$, if $p \leq p'$ and $p' \leq p''$, then $p \leq p''$.

A *partially ordered set* is a pair $(P, \leq)$, in which P is a set and $\leq$ is a partial order on P.

The main partially ordered sets we will consider are the power set 2^X of a given set X with the relation $\subseteq$ of set containment, and the subset of open sets $\mathcal{T} \subseteq 2^X$ of a topological space $(X, \mathcal{T})$, also with the containment relation.

Definition 2.2. Let $(P, \leq)$ be a partially ordered set. An *ideal* of $(P, \leq)$ is a set $I \subseteq P$ such that

1. (Non-triviality) $I \neq \emptyset$.
2. (Downward closure) For every $i \in I$ and $p \in P$, if $p \leq i$, then $p \in I$.
3. (Upward directed) For every $i_0, i_1 \in I$, there exists a $i \in I$ such that $i_0 \leq i$ and $i_1 \leq i$.

A *σ-ideal* of $(P, \leq)$ is an ideal $I \subseteq P$ such that

4. (Upward σ-directed) For every sequence $(i_n)_{n \in \mathbb{N}}$ in I, there exists a $i \in I$ such that, for every $n \in \mathbb{N}$, $i_n \leq i$.

The prototypes of this construction of order theory ideals are the ideals in ring theory, which generalize the 'negligible' behavior that the number 0 has for the integers. As an example, we can compare the behavior of 0 with the set of even numbers: in the same way that $0 + 0 = 0$ and $0 \times n = 0$ for every integer n, it is also true that the sum of two even numbers is an even number, and that the product of an even number by any number is also an even number. Ideals in ring theory are related to the order theory ideals by the divisibility relation. The dual notion of an ideal is a filter.

Definition 2.3. Let $(P, \leq)$ be a partially ordered set. A *filter* of $(P, \leq)$ is a set $F \subseteq P$ such that

1. (Non-triviality) $F \neq \emptyset$.
2. (Upward closure) For every $f \in F$ and $p \in P$, if $f \leq p$, then $p \in F$.
3. (Downward directed) For every $f_0, f_1 \in F$, there exists a $f \in F$ such that $f \leq f_0$ and $f \leq f_1$.

A *σ-filter* of $(P, \leq)$ is a filter $F \subseteq P$ such that

4. (Downward σ-directed) For every sequence $(f_n)_{n \in \mathbb{N}}$ in F, there exists a $f \in I$ such that, for every $n \in \mathbb{N}$, $f \leq f_n$.

2.1.2 Codense and dense sets

The context is a general topological space $\boldsymbol{X} = (X, \mathcal{T})$. Remember that we denote the topological interior of a set $S \subseteq X$ by S° and its topological closure by $S^\bullet$.

Definition 2.4. Let $\boldsymbol{X}$ be a topological space.

1. A *codense* set is a set $D \subseteq X$ such that $D^\circ = \emptyset$.
2. A *dense* set is a set $D \subseteq X$ such that $D^\bullet = X$.

The intersection of two open sets may fail to be dense. For instance, the rational numbers and the irrational numbers are both dense in the real line, but their intersection is empty. Likewise, the union of two codense sets may not be codense. Nevertheless, if we consider only open dense sets — or closed codense sets — these properties are in fact true.

Proposition 2.1. *Let $\boldsymbol{X}$ be a topological space. The open dense sets form a filter of $(\mathcal{T}, \subseteq)$, and the closed codense sets form an ideal of the closed sets with containment order.*

Proof. We will only prove the proposition for open dense sets, since it follows by duality for closed codense sets.

1. (Non-triviality) The set of open dense sets is not empty since X is an open dense set.
2. (Upward closure) Let $D \subseteq X$ be an open dense set and $D \subseteq S$ an open set. Then $X = D^{\bullet} \subseteq S^{\bullet}$, hence $X = S^{\bullet}$, which shows that S is dense.
3. (Downward directed) Let $D_0, D_1 \subseteq X$ be open dense sets. We will show that their intersection $D_0 \cap D_1$ is also dense, since it is open by definition of the topology. Let $U \subseteq X$ be an non-empty open set. Since D_0 is open, then $V \cap D_0$ is open and, since D_0 is dense, then $V \cap D_0$ is non-empty. Now, since D_1 is dense, it follows that $(V \cap D_0) \cap D_1$ is non-empty. This shows that the intersection $D_0 \cap D_1$ is dense, because its intersection with any non-empty open set is non-empty. (In fact, it is sufficient that one of the dense sets be open (D_0 in our proof) in order to guarantee that their intersection be dense.) ■

This shows that the dense sets form an ideal in the topology $\mathcal{T}$ of the space $\boldsymbol{X}$, but not in its power set 2^X. To obtain that, we must broaden our definitions.

2.1.3 Rare and corare sets

We are going to consider sets whose closure is codense. These sets include closed codense sets, but also sets which are not closed. Dually, we may also consider sets whose interior is dense. Remember that we denote the topological interior of a set $S \subseteq X$ by S° and its topological closure by $S^{\bullet}$, and its set complement by $\overline{S}$.

Definition 2.5. Let $\boldsymbol{X}$ be a topological space.

1. A *rare* (or *nowhere dense*) set is a set $R \subseteq X$ such that $(R^{\bullet})^{\circ} = \mathbb{0}$.
2. A *corare* set is a set $R \subseteq X$ such that $(R^{\circ})^{\bullet} = X$.

Rare sets are also called nowhere dense sets because of the following property, which we leave as an exercise.

Exercise 2.1. Let $\boldsymbol{X}$ be a topological space. A set $R \subseteq X$ is rare if, and only if, for every non-empty set $S \subseteq X$, the intersection $S \cap R$ is not dense in S.

Proposition 2.2. *Let $\boldsymbol{X}$ be a topological space. The rare sets form an ideal of $(2^X, \subseteq)$, and the corare sets form a filter of $(2^X, \subseteq)$.*

Proof. We will prove the proposition for rare sets.

1. (Non-triviality) The empty set $\mathbb{0}$ is rare.
2. (Downward closure) Let $R \subseteq X$ be a rare set and $S \subseteq R$. Then $S^{\bullet} \subseteq R^{\bullet}$, hence $(S^{\bullet})^{\circ} \subseteq (R^{\bullet})^{\circ} = \mathbb{0}$, so it follows that $(S^{\bullet})^{\circ} = \mathbb{0}$, which shows that S is rare.
3. (Upward directed) Let $R_0, R_1 \subseteq X$ be rare sets. From properties of closure, $(R_0 \cup R_1)^{\bullet} = R_0^{\bullet} \cup R_1^{\bullet}$. To prove that $((R_0 \cup R_1)^{\bullet})^{\circ} = \mathbb{0}$, let $U \subseteq (R_0 \cup R_1)^{\bullet}$ be an open set. We will show that U is empty. Let $V \coloneqq U \cap \overline{R_0^{\bullet}}$. By definition, $V \subseteq R_1^{\bullet}$. This set V is open because U is open and $\overline{R_0^{\bullet}}$ is the complement of a closed set, hence open. For the sake of contradiction, suppose that U were non-empty. Then V would also be non-empty, since otherwise $U \subseteq R_0^{\bullet}$, which would contradict the fact that R_0 is rare (every open set contained in its closure is empty). Hence V would be a non-empty open set such that $V \subseteq R_1^{\bullet}$, which would contradict the fact that R_1 is rare. Therefore U must be empty, which shows that $R_0 \cup R_1$ is rare. ■

The fact that the union of two rare sets is also rare can be generalized by induction to any finite union of rare sets, but this is not true in general for countable unions. For instance, the rational numbers are a countable union of rare sets (each rational point on the real line), but they are dense, hence not rare. This means that we cannot guarantee the rare sets form a σ-ideal. Again, we must broaden our definitions to have such property.

2.1.4 Meager and comeager sets

The sets we will consider now are simply countable unions of rare sets. This forces them to be a σ-ideal. Remember that we denote the topological interior of a set $S \subseteq X$ by S° and its topological closure by $S^\bullet$.

Definition 2.6. Let X be a topological space.

1. A *meager* set is a set $M \subseteq X$ for which there exists a countable collection $(R_n)_{n\in\mathbb{N}}$ of rare sets such that
$$M = \bigcup_{n\in\mathbb{N}} R_n.$$
A *nonmeager* set is a set that is not meager.
2. A *comeager* (or *residual*) set is a set $M \subseteq X$ for which there exists a countable collection $(R_n)_{n\in\mathbb{N}}$ of corare sets such that
$$M = \bigcap_{n\in\mathbb{N}} R_n.$$

Proposition 2.3. *Let X be a topological space. The meager sets form a σ-ideal of $(2^X, \subseteq)$, and the comeager sets form a σ-filter of $(2^X, \subseteq)$.*

Proof. We will prove the proposition for meager sets.

1. (Non-triviality) The empty set $\emptyset$ is meager.
2. (Downward closure) Let $M \subseteq X$ be a meager set and $S \subseteq M$. Since M is meager, there exists a sequence $(R_n)_{n\in\mathbb{N}}$ of rare sets such that $M = \bigcup_{n\in\mathbb{N}} R_n$. For each $n \in \mathbb{N}$, the set $S \cap R_n$ is rare, since $((S \cap R_n)^\bullet)^\circ \subseteq (R_n^\bullet)^\circ = \emptyset$. Then from
$$S = S \cap M = \bigcup_{n\in\mathbb{N}} S \cap R_n$$
it follows that S is meager.
3. (Upward σ-directed) Let $M_0, M_1, \ldots \subseteq X$ be a sequence of meager sets. Then, for each $n \in \mathbb{N}$, there exists a sequence $(R_{n,m})_{m\in\mathbb{N}}$ of rare sets such that $M_n = \bigcup_{m\in\mathbb{N}} R_{n,m}$. So it follows that
$$M_\cup := \bigcup_{n\in\mathbb{N}} M_n = \bigcup_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} R_{n,m},$$
which shows that $M_\cup$ is a countable union rare sets, hence meager. ∎

2.1.5 Almost open sets

This is based on [Tao09b].

Definition 2.7. Let X be a topological space. An *almost open* set is a set $A \subseteq X$ for which there exists an open set $U \subseteq X$ such that the symmetric difference $A \triangle U$ is meager.

Question 2.1. Let X be a topological space. Are the almost open sets an ideal/σ-ideal? Filter?

2.2 Baire spaces and the relationship between dense and comeager sets

A Baire space is a space where the notion of being meager implies the notion of being codense, or, equivalently, where the notion of being comeager (residual) implies the notion of being dense. These spaces are special since these two different notions of negligibility are not always equivalent.

Definition 2.8. A *Baire space* is a topological space $\mathbf{X}$ in which every meager set is codense.

We could use any of the properties from PROPOSITION 2.4 to define Baire spaces. We choose PROPERTY 1 because it is the most meaningful in the context we are presenting about the different types of set negligibility, but the most commonly used one to define Baire spaces is PROPERTY 4.

In the following proposition we show the equivalence of many notions relating meager and comeager sets to codense and dense sets. Remember that we denote the topological interior of a set $S \subseteq X$ by S° and its topological closure by $S^\bullet$.

Proposition 2.4. *Let $\mathbf{X}$ be a topological space. The following properties of $\mathbf{X}$ are equivalent to each other.*

1. *Every meager set is codense (Baire space).*
2. *Every comeager set is dense.*
3. *Every countable union of closed codense sets is codense.*
4. *Every countable intersection of open dense sets is dense.*
5. *Every non-empty open set is nonmeager.*

Proof. PROPERTIES 1 and 2 are equivalent because a set is dense if, and only if, its complement is codense, and it is meager if, and only if, its complement is comeager. PROPERTIES 3 and 4 are also equivalent using complements.

Let us show that PROPERTIES 1 and 3 are equivalent. Assume PROPERTY 1 is true. Since every closed codense set is a rare set, a countable union of closed codense sets is meager, therefore is codense by PROPERTY 1. Now assume PROPERTY 3 is true, and let $M \subseteq X$ be a meager set and $(R_n)_{n\in\mathbb{N}}$ a countable collection of rare sets such that $M = \bigcup_{n\in\mathbb{N}} R_n$. Since each R_n is rare, we have $(R_n^\bullet)^\circ = \emptyset$, so its closure $R_n^\bullet$ is codense. Then from PROPERTY 3 it follows that $M^\bullet = \bigcup_{n\in\mathbb{N}} R_n^\bullet$ is a codense set because it is a union of closed codense sets. This means that $(R_n^\bullet)^\circ = \emptyset$, therefore $M^\circ \subseteq (M^\bullet)^\circ = \emptyset$, which shows that the meager set M is codense.

Finally, let us show that PROPERTIES 1 and 5 are equivalent. Assuming PROPERTY 1, if U is open and meager, then its interior is empty, so it is empty. Conversely, assuming PROPERTY 5, for each meager set M, its interior M° is also meager, since it is a subset of a meager set (PROPOSITION 2.3). However, M° is also open, so from PROPERTY 5 it must be empty, that is, $M^\circ = \emptyset$, which means that the meager set M is codense. ■

The Baire category theorem (THEOREM 2.5) is a classical result that provides two different sufficient conditions a space may satisfy in order for it to be a Baire space. Neither condition implies the other, since there are locally compact metric spaces that are not completely metrizable and vice-versa. The following demonstration is based on Rudin [Rud91, §2.2].

Theorem 2.5 (Baire)**.** *Let $\mathbf{X}$ be a topological space.*

1. *If $\mathbf{X}$ is a locally compact Hausdorff space, then it is a Baire space.*
2. *If $\mathbf{X}$ is a completely metrizable space, then it is a Baire space.*

Proof. We present each proof separately, but they are essentially very similar. Let $(D_n)_{n\in\mathbb{N}}$ be a countable collection of open dense sets and $U \subseteq X$ be a non-empty open set. We will prove that $D := \bigcap_{n\in\mathbb{N}} D_n$ is dense by showing that $D \cap U \neq \emptyset$.

1. Suppose $\boldsymbol{X}$ is a locally compact Hausdorff space. This implies that every point of X has a local basis of compact neighborhoods. Since D_0 is dense and U is open, there exists $p_0 \in D_n \cap U$, and, since D_0 is open, the intersection $D_0 \cap U$ is open, so because every point of X has a local basis of compact neighborhoods, it follows that there exists a compact set $K_0 \subseteq D_0 \cap U$. Now we recursively define, for every $n \in \mathbb{N}$, a compact set K_{n+1} such that
$$K_{n+1} \subseteq D_{n+1} \cap K_n.$$
It then follows from PROPOSITION 1.21 that $\bigcap K_n$ is compact, hence $\bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$. But notice that, for every $n \in \mathbb{N}$, $K_n \subseteq D_n \cap U$, because $K_n \subseteq D_n$ and, by induction, $K_n \subseteq K_{n-1} \subseteq \cdots \subseteq K_0 \subseteq U$, so
$$\emptyset \neq \bigcap_{n\in\mathbb{N}} K_n \subseteq \bigcap_{n\in\mathbb{N}} D_n \cap U = D \cap U,$$
hence $D \cap U \neq \emptyset$, which shows that D is dense.
2. Suppose $\boldsymbol{X}$ is completely metrizable and $|\cdot,\cdot|$ is a metric on X that makes $\boldsymbol{X}$ a complete metric space. Since D_0 is dense and U is open, there exists $p_0 \in D_0 \cap U$, and, since D_0 is open, the intersection $D_0 \cap U$ is open, so there exists a real number $0 < r_0 < 1$ such that $\mathrm{B}[p_0; r_0] \subseteq D_0 \cap U$. Now we recursively define sequences $(p_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ as follows. Denote $B_n := \mathrm{B}(p_n; r_n)$ Given $n \in \mathbb{N}$, the intersection $D_{n+1} \cap B_n$ is non-empty since D_{n+1} is dense, thus there exists a point $p_{n+1} \in D_{n+1} \cap B_n$. Also, since $D_{n+1} \cap B_n$ is open, there exists a real number $0 < r_{n+1} < 2^{-(n+1)}$ such that
$$B^\bullet_{n+1} \subseteq D_{n+1} \cap B_n.$$
It then follows from PROPOSITION 1.32 that $\bigcap_{n\in\mathbb{N}} B^\bullet_n \neq \emptyset$. But notice that, for every $n \in \mathbb{N}$, $B^\bullet_n \subseteq D_n \cap U$, because $B^\bullet_n \subseteq D_n$ and, by induction, $B^\bullet_n \subseteq B_{n-1} \subseteq \cdots \subseteq B_0 \subseteq U$, so
$$\emptyset \neq \bigcap_{n\in\mathbb{N}} B^\bullet_n \subseteq \bigcap_{n\in\mathbb{N}} D_n \cap U = D \cap U,$$
hence $D \cap U \neq \emptyset$, which shows that D is dense. ■

Exercise 2.2. Show that the irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ are a Baire space.

2.3 G_δ and F_σ sets

G_δ and F_σ sets are related to the logical structure used to define them. G_δ sets are sets which can be defined by a countable universal quantifier over 'open conditions'. Dually, F_σ sets are defined by a countable existential quantifier over 'closed conditions'. For instance, since being a point is a closed condition (a singleton is closed in the line $\mathbb{R}$), and there are countable rational numbers, $\mathbb{Q}$ is an F_σ set. The irrational numbers are therefore a G_δ : not being a point is an open condition (the complement of a singleton is open in $\mathbb{R}$) and an irrational point is a point that is not any rational number, so a countable universal quantifier is used. These examples will become clearer after the definition.

Definition 2.9. Let $\boldsymbol{X}$ be a topological space.

1. A *G_δ set* is a set $G \subseteq X$ for which there exists a countable collection $(A_n)_{n\in\mathbb{N}}$ of open sets such that
$$G = \bigcap_{n\in\mathbb{N}} A_n.$$
2. An *F_σ set* is a set $F \subseteq X$ for which there exists a countable collection $(F_n)_{n\in\mathbb{N}}$ of closed sets such that
$$F = \bigcup_{n\in\mathbb{N}} F_n.$$

We first state some basic consequences of the definitions.

Proposition 2.6. *Let $\boldsymbol{X}$ be a topological space.*

1. *An open set is a G_δ set.*
2. *A closed set is an F_σ set.*
3. *The intersection of a countable collection of G_δ sets is a G_δ set.*
4. *The union of a finite collection of G_δ sets is a G_δ set.*
5. *The union of a countable collection of F_σ sets is a F_σ set.*
6. *The intersection of a finite collection of F_σ sets is a F_σ set.*
7. *The complement of a G_δ set is an F_σ set.*

Example 2.1. The rational numbers $\mathbb{Q}$ are an F_σ set in the real numbers $\mathbb{R}$, because they are countable and, for each $q \in \mathbb{Q}$, the singleton set $\{q\}$ is closed. More generrally, every countable set in an accessible topological space is an F_σ set.

Example 2.2. The irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ are a G_δ set in the real numbers $\mathbb{R}$, because they are the complement of the rational numbers, which are an F_σ set. Explicitely, the irrational numbers can be given as $\mathbb{R} \setminus \mathbb{Q} = \bigcap_{q\in\mathbb{Q}} \overline{\{q\}}$, each set $\overline{\{q\}}$ being not only open but also dense, hence corare. This shows that the irrational numbers are a comeager (residual) set.

Example 2.3. The rational numbers are not a G_δ set. If they were, they would be a contable intersection of open sets $(A_n)_{n\in\mathbb{N}}$, but then each A_n would also be dense in $\mathbb{R}$, because $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{Q} \subseteq A_n$. Since the irrational numbers are a countable intersection of open dense sets (complements of rational points), it would be possible to express the $\mathbb{0}$, which is the intersection of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$, as a countable intersection of open dense sets. This is in contradiction with the fact that the real numbers is a Baire space (as a consequence of THEOREM 2.5), because in a Baire space every countable intersection of open dense sets is dense in the space (PROPOSITION 2.4).

For the next proposition, remember that a continuity point of a function $f : X \to X'$ between topological spaces $\boldsymbol{X}$ and $\boldsymbol{X}'$ is a point $x \in X$ for which, given an neighborhood V' of $f(x) \in X'$, there exists a neighborhood V of x such that $f(V) \subseteq V'$. When $\boldsymbol{X}' = \boldsymbol{M}$ is a metric space, this is equivalent to the following: for every $\varepsilon \in \mathbb{R}_{>0}$, there exists a neighborhood V of x such that $f(V) \subseteq \mathrm{B}(f(x); \varepsilon)$; that is, for every $x' \in V$, $|f(x), f(x')| < \varepsilon$.

Proposition 2.7. *Let $\boldsymbol{X}$ be a topological space, $\boldsymbol{M}$ a metrizable topological space and $f : X \to M$ a function. The continuity set C of f (the set of points $C \subseteq X$ on which f is continuous) is a G_δ set and the discontinuity set D of f (the set of points $D \subseteq X$ on which f is not continuous) is an F_σ set.*

Proof. Let $|\cdot,\cdot|$ be a metric that generates the topology of $\boldsymbol{M}$. For each $n \in \mathbb{Z}_{>0}$, define the set

$$A_n := \left\{ x \in X \;\middle|\; \text{there exists a neigborhood } V \text{ of } x \text{ such that, for every } x', x'' \in V, |f(x'), f(x'')| < \frac{1}{n} \right\}.$$

We first show that the set A_n is open. Let $x \in A_n$ and take a neighborhood $V \subseteq X$ of x such that, for every $x', x'' \in V$, it holds that $|f(x'), f(x'')| < \frac{1}{n}$. For every $y \in V$, the set V is also a neighborhood of y and it has the desired property for y to be an element of A_n. Then $V \subseteq A_n$, which shows that A_n is open.

Now define the G_δ set $G := \bigcap_{n\in\mathbb{Z}_{>0}} A_n$. We will prove that $G = C$.

- $(C \subseteq G)$ Let $x \in C$ and take $n \in \mathbb{Z}_{>0}$. Since x is in the continuity set of f, there exists a neighborhood $V \subseteq X$ of x such that, for every $x' \in V$, $|f(x), f(x')| < \frac{1}{2n}$. Then it follows by the triangle inequality that, for every $x', x'' \in V$,

$$|f(x'), f(x'')| \leq |f(x'), f(x)| + |f(x), f(x'')| < \frac{1}{n},$$

which shows that $x \in A_n$ and hence that $x \in G$.

- ($G \subseteq C$) Let $x \in G$ and take $\varepsilon \in \mathbb{R}_{>0}$. There exists $n \in \mathbb{Z}_{>0}$ such that $\frac{1}{n} \leq \varepsilon$, and, since $x \in A_n$, there exists a neighborhood $V \subseteq X$ of x such that, for all $x' \in V$,

$$|f(x), f(x')| < \frac{1}{n} \leq \varepsilon,$$

which shows that x is a continuity point of f, so $x \in C$.

This shows that C is a G_δ set and the fact that the discontinuity set is an F_σ set then follows by set complementation. ■

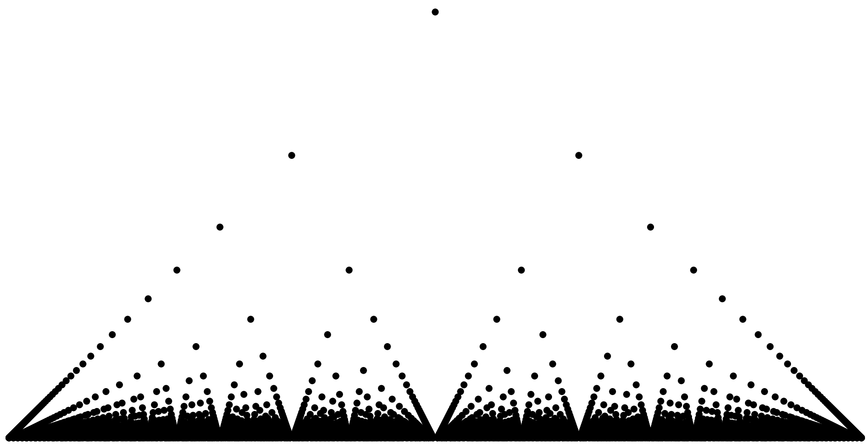

Figure 2. Thomae's function is a function that is only continuous at irrational points.

Proposition 2.8. *A subspace of a completely metrizable space* $\boldsymbol{M}$ *is itself completely metrizable if, and only if, it is a* G_δ *set in* $\boldsymbol{M}$.

2.4 Measurably negligible sets

This is based on [Tao09b].

Denote $M \doteq M'$ for $m((M \setminus M') \cup (M' \setminus M))$.

Definition 2.10. Let $\boldsymbol{X} = (X, \mathcal{M}, m)$ be a measure space.

1. A *null* set is a measurable set $N \in \mathcal{M}$ such that $N \doteq \emptyset$ (equivalently, $m(N) = 0$).
2. A *conull* (or *full measure*) set is a measurable set $N \in \mathcal{M}$ such that $N \doteq X$ (equivalently, $m(N) = m(X)$).

Proposition 2.9. *Let* $\boldsymbol{X}$ *be a measure space. The null sets form a σ-ideal of* $(\mathcal{M}, \subseteq)$, *and the conull sets form a σ-filter of* $(\mathcal{M}, \subseteq)$.

Proof. We will prove the proposition for null sets.

1. (Non-triviality) The empty set $\emptyset$ is null.
2. (Downward closure) Let $N \subseteq X$ be a null set and $S \subseteq N$ a measurable set. Then $m(S) \leq m(N) = 0$, which shows that S is null.

3. (Upward σ-directed) Let $N_0, N_1, \ldots \subseteq X$ be a sequence of null sets and $N_\cup := \bigcup_{n\in\mathbb{N}} N_n$. Then $N_\cup$ is measurable and

$$m(N_\cup) = m\left(\bigcup_{n\in\mathbb{N}} N_n\right) \leq \sum_{n\in\mathbb{N}} m(N_n) = 0,$$

which shows that $N_\cup$ is null. ■

3 Topological dynamics

3.1 Topological transitivity

The simplest notion of transitivity comes from group actions.

Definition 3.1. Let X be a set and $\boldsymbol{G}$ a group. A *transitive* group action of $\boldsymbol{G}$ on X is a group action $a\colon G \times X \to X$ for which, given any pair of points $x, x' \subseteq X$, there exists $g \in G$ such that $a^g(x) = x'$.

Thus, in the context discrete-time invertible dynamics, we could consider the definition group actions of $\mathbb{Z}$: A *transitive* dynamics on X is an invertible dynamics $f\colon X \to X$ for which, given any pair of points $x, x' \subseteq X$, there exists an integer $n \in \mathbb{Z}$ such that $f^n(x) = x'$. This is equivalent to the existence of an orbit of a point of the system that equals the whole space X.

If we restrict this to dynamics that are not necessarily invertible, we must consider semigroup actions of $\mathbb{Z}_{\geq 0}$, and we obtain the following definition: A *transitive* dynamics on X is a dynamics $f\colon X \to X$ for which, given any pair of points $x, x' \subseteq X$, there exists a positive integer $n \in \mathbb{Z}_{\geq 0}$ such that $f^n(x) = x'$. This in turn is equivalent to the existence of a forward orbit of a point of the system that equals X, in fact a periodic orbit, which implies that ths system is in fact invertible.

This definition is, of course, very restrictive for our considerations. When we consider a topological space instead of a set, we can weaken this notion of transitivity. A point does not necessarily need to reach another point, only come arbitrarily close to it. This means it needs to visit every open neighborhood of the other points in the space or, in other words, its forward orbit must be dense. Because of its direct relation to the concept of transitivity, we call this *topological transitivity*, but this is often simply called *transitivity* in the context of topological dynamics, in which the much stronger notion of transitivity of (semi)group actions is barely useful.

Definition 3.2. Let $\boldsymbol{X}$ be a topological space. A *topologically transitive* dynamics on X is a continuous dynamics $f\colon X \to X$ for which there exists a point $x \in X$ whose forward orbit $f^{\mathbb{Z}_{\geq 0}}(x)$ is dense in X. Such a point is called a *topologically transitive* point. The *transitive set* of f, denoted $\mathrm{T}(f)$, is the set of all topologically transitive points of f.

Under a different perspective, instead of considering a point that visits every neighborhood, we can consider dynamics in which (non-empty) neighborhoods of the space visit each other. This gives rise to another definition, which we call *regional topological transitivity* following Gottschalk and Hedlund [GH55], but which is often called *topological transitivity*.

Definition 3.3. Let $\boldsymbol{X}$ be a topological space. A *regionally (topologically) transitive* dynamics on X is a continuous dynamics $f\colon X \to X$ for which, given any pair of non-empty open sets $U, U' \subseteq X$, there exists a positive integer $n \in \mathbb{Z}_{\geq 0}$ such that $f^n(U) \cap U' \neq \emptyset$.

There are other, more general notions of topological transitivity which distinguish forward and backward orbits, or take into account hitting sets etc. Akin and Carlson present a comprehensive discussion of these definitions and their relations [AC12]. Gottschalk and Hedlund present a discussion about topological transitivity for flows and, more generally, for (semi)group actions [GH55].

The concepts of topological transitivity and regional transitivity presented in DEFINITIONS 3.2 and 3.3 are not equivalent for arbitrary topological spaces. In fact, neither implies the other. Nevertheless, under some reasonable assumptions on the topological space, the two notions are equivalent. The classical result in this regard is Birkhoff's transitivity theorem.

Theorem 3.1 (Birkhoff). *Let $\boldsymbol{M}$ be a perfect, second-countable and complete metric space. Then a continuous dynamics $f : M \to M$ is regionally transitive if, and only if, it is topologically transitive.*

In what follows, we will prove two propositions, PROPOSITIONS 3.3 and 3.5, which, along with the fact that every metric space is accessible and every complete metric space is Baire, imply THEOREM 3.1. Each of these propositions consider different sufficient conditions the topological space must satisfy in order for one notion of topological transitivity to imply the other. To prove the first one, we need a simple lemma.

Lemma 3.2. *Let $\boldsymbol{X}$ be an accessible and perfect topological space and $f : X \to X$ a continuous dynamics. If a point $x \in X$ is topologically transitive, then, for every positive integer $n \in \mathbb{Z}_{\geq 0}$, the point $f^n(x)$ is also topologically transitive.*

Proof. Let $x \in X$ be a topologically transitive point and take a positive integer $n \in \mathbb{Z}_{\geq 0}$ and an non-empty open set $U \subseteq X$. Consider the set $f^{[\![n]\!]}(x) = \{f^m(x) \mid 0 \leq m < n\}$ and define $U' := U \setminus f^{[\![n]\!]}(x)$. Since U is a non-empty open set, $f^{[\![n]\!]}(x)$ is a finite set, and $\boldsymbol{X}$ accessible and perfect, the set U' is a non-empty open set (LEMMA 1.2).

So the set U' is a non-empty open set that does not contain any of the $f^m(x)$ for $0 \leq m < n$. Since x is topologically transitive, it follows that there exists $n' \geq 0$ such that $f^{n'}(x) \in U'$. But the definition of U' implies that $n' \geq n$ and, since $U' \subseteq U$, if follows that $f^n(x)$ is also topologically transitive. ■

Proposition 3.3. *Let $\boldsymbol{X}$ be an accessible and perfect topological space and $f : X \to X$ a continuous dynamics. If f is topologically transitive, then it is regionally transitive.*

Proof. Let $x \in X$ be a topologically transitive point and take a pair of non-empty open sets $U, U' \subseteq X$. Since x is topologically transitive, there exists a positive integer $m \geq 0$ such that $f^m(x) \in U$. Because the space $\boldsymbol{X}$ is perfect, the point $f^m(x)$ is also topologically transitive (LEMMA 3.2), so there exists a positive integer $n \geq 0$ such that $f^n(f^m(x)) \in U'$. It follows that $f^{m+n}(x) \in f^n(U) \cap U'$, therefore $f^n(U) \cap U' \neq \emptyset$, which shows that f is regionally transitive. ■

Wrong proof. Where is the mistake?[2] Let $x \in X$ be a topologically transitive point and take a pair of non-empty open sets $U, U' \subseteq X$. Since x is topologically transitive, there exists a positive integer $n \geq 0$ such that $f^n(x) \in U'$ and, because f is continuous, $f^{-n}(U')$ is an open set, hence $U \cap f^{-n}(U')$ is also open. Since x is topologically transitive, there exists a positive integer $m \geq 0$ such that $f^m(x) \in U \cap f^{-n}(U')$. It follows that $f^{m+n}(x) \in f^n(U) \cap U'$, therefore $f^n(U) \cap U' \neq \emptyset$, which shows that f is regionally transitive. ■

Before proving PROPOSITION 3.5, we provide a usefull classification of the set of topologically transitive points in second-countable spaces.

Lemma 3.4. *Let $\boldsymbol{X}$ be a second-countable topological space and $f : X \to X$ a continuous dynamics. The set $\mathrm{T}(f)$ of topologically transitive points of f is a G_δ set. If $(U_n)_{n \in \mathbb{N}}$ is a countable basis of non-empty open sets for the topology of $\boldsymbol{X}$, then, for every $n \in \mathbb{N}$, the union $\bigcup_{m=0}^{\infty} f^{-m}(U_n)$ is a non-empty open set and*

$$\mathrm{T}(f) = \bigcap_{n=0}^{\infty} \bigcup_{m=0}^{\infty} f^{-m}(U_n).$$

2. The mistake is in assuming that $U \cap f^{-n}(U')$ is non-empty to be able to use the topological transitivity of the point x, since this is exactly what we are trying to prove. This "proof" was suggested by a Large Language Model after some even more wrong attempts at proving this proposition.

Proof. Let $(U_n)_{n\in\mathbb{N}}$ be a countable basis of non-empty open sets for the topology of $\boldsymbol{X}$. For each $n \in \mathbb{Z}_{\geq 0}$, the set of points of X that eventually visit U_n in positive time under the action of f is the set

$$V_n := \bigcup_{m=0}^{\infty} f^{-m}(U_n).$$

Since U_n is open and f is continuous, V_n is also open, and it is non-empty since $\emptyset \neq U_n \subseteq V_n$.

Finally, let us consider the set

$$T := \bigcap_{n=0}^{\infty} V_n = \bigcap_{n=0}^{\infty} \bigcup_{m=0}^{\infty} f^{-m}(U_n).$$

This is the set of points of X that eventually visit each open basic set U_n in positive time under the action of f or, equivalently, the set of all topologically transitive points of the system. This also shows that the set of topologically transitive points is a G_δ set. ∎

For the next proposition, let's remember that in a Baire space every countable intersection of open dense sets is a dense set (PROPOSITION 2.4).

Proposition 3.5. *Let $\boldsymbol{X}$ be a second-countable and Baire topological space and $f: X \to X$ a continuous dynamics. If f is regionally transitive, then it is topologically transitive.*

Proof. Let $(U_n)_{n\in\mathbb{N}}$ be a countable basis of non-empty open sets for the topology of $\boldsymbol{X}$. For each $n \in \mathbb{Z}_{\geq 0}$, define

$$D_n := \bigcup_{m=0}^{\infty} f^{-m}(U_n).$$

From LEMMA 3.4, we know that each D_n is a non-empty open set and that the transitive set of f is given by

$$\mathrm{T}(f) = \bigcap_{n=0}^{\infty} D_n.$$

We will show that each D_n is dense in X. Let $V \subseteq X$ be a non-empty open set. Since f is regionally transitive, for the non-empty open sets U_n and V there exists a positive integer $m \geq 0$ such that $f^m(V) \cap U_n \neq \emptyset$, which is equivalent to $V \cap f^{-m}(U_n) \neq \emptyset$. This implies that $V \cap D_n \neq \emptyset$, which shows that D_n is dense in X.

Finally, since $\boldsymbol{X}$ is a Baire space and the sets D_n are open and dense, it follows that D is also dense, which implies that it is not empty (the case in which $X = \emptyset$ is trivial), so there exists a topologically transitive point and hence f is topologically transitive. ∎

3.1.1 Topological transitivity for forward and backward orbits

Proposition 3.6. *Let $\boldsymbol{X}$ be a perfect and compact metric space and $f: X \to X$ a homeomorphism. If there exists a point $x \in X$ whose orbit $f^{\mathbb{Z}}(x)$ is dense in X, then f is topologically transitive.*

Proof. This uses PROPOSITION 3.5. Check [MSE18]. ∎

Proposition 3.7. *Let $\boldsymbol{X}$ be a perfect and compact metric space and $f: X \to X$ a homeomorphism. If f is topologically transitive, then f^{-1} is topologically transitive.*

Proof. This uses PROPOSITION 3.5. Check [MSE18].

∎

3.2 Recurrence

Definition 3.4. Let $\boldsymbol{X}$ be a topological space and $f: X \to X$ a continuous dynamics. A *recurrent* point of f is a point $x \in X$ such that, for every neighborhood V of x, there exists $n \in \mathbb{Z}_{>0}$ such that $f^n(x) \in V$. The *recurrent* set of f is the set of all recurrent points of f, denoted $\mathrm{R}(f)$.

Other definitions of recurrent point exists and they are all equivalent (see [LS15]). For instance, this is equivalent to the existence of a sequence $(n_i)_{i\in\mathbb{N}}$ of natural numbers such that $\lim_{i\to\infty} n_i = \infty$ and $\lim_{i\in\mathbb{N}} f^{n_i} = x$. The following proof is based on [VO16].

Proposition 3.8 (Birkhoff recurrence theorem)**.** *Let $\boldsymbol{X}$ be a compact topological space and $f: X \to X$ a continuous dynamics. Then*

$$\mathrm{R}(f) \neq \emptyset.$$

Proof. Let $\mathcal{I} \subseteq 2^X$ be the set of all closed non-empty forward invariant sets $I \subseteq X$ (that is, $f(I) \subseteq I$). This family is non-empty because $X \in \mathcal{I}$. Let us show that a set $I \in \mathcal{I}$ is minimal with respect to the containment relation $\subseteq$ if, and only if, the orbit of every $x \in I$ is dense in I.

- ($\Rightarrow$) Let $I \in \mathcal{I}$ be minimal with respect to the containment relation $\subseteq$. Since I is forward invariant, the orbit of every $x \in I$ in included in I, and, since I is closed, the topological closure of such orbit is contained in I. So, because I is minimal, the closure of this orbit must equal I, otherwise the closure of the orbit would be a set $C \subset I$ that is closed, non-empty and forward invariant, contradicting the minimality of I.
- ($\Leftarrow$) Let $I \in \mathcal{I}$ be a set such that the orbit of every $x \in I$ is dense in I. Then every closed, non-empty and forward invariant set $C \subseteq I$ must equal I, because closure of the orbit of every $x \in C$ is contained in C and equals I. This shows that I is minimal.

This implies that every point in such a minimal set is recurrent, so it is enough to prove there exists a minimal set.

To this end we show that every decreasing chain $(I_\omega)_{\omega\in\Omega}$ in $\mathcal{I}$ has a minorant. Let $I_\cap := \bigcap_{\omega\in\Omega} I_\omega$. Since $(I_\omega)_{\omega\in\Omega}$ is a decreasing chain of closed non-empty sets and $\boldsymbol{X}$ is compact, their intersection $I_\cap$ is non-empty (PROPOSITION 1.13). The intersection $I_\cap$ is closed because it is an intersection of closed sets I_ω, and is forward invariant because each I_ω is forward invariant, so $f(I_\cap) = \bigcap_{\omega\in\Omega} f(I_\omega) \subseteq I_\cap$. Since $I_\cap \subseteq I_\omega$ for every I_ω, this shows $I_\cap$ is a minorant of $(I_\omega)_{\omega\in\Omega}$. Finally, it then follows from Zorn's lemma that $\mathcal{I}$ contains a minimal element, hence f has a recurrent point. ∎

The next result shows that the recurrent set is a G_δ set.

Proposition 3.9. *Let $\boldsymbol{X}$ be a metric space and $f: X \to X$ a continuous dynamics. The recurrent set $\mathrm{R}(f)$ is a G_δ set and*

$$\mathrm{R}(f) = \bigcap_{m\geq 1} \bigcup_{n\geq m} \left\{ x \in X \,\middle|\, |x, f^n(x)| < \frac{1}{m} \right\}.$$

Proof. For every $m, n \in \mathbb{Z}_{>0}$ such that $n \geq m \geq 1$, define the sets

$$R_{m,n} := \left\{ x \in X \,\middle|\, |x, f^n(x)| < \frac{1}{m} \right\},$$

$R_m := \bigcup_{n\geq m} R_{m,n}$ and $R := \bigcap_{m\geq 1} R_m$. The function $d: X \to \mathbb{R}_{\geq 0}$ defined by $d(x) := |x, f(x)|$ is continuous, because f and $|\cdot,\cdot|$ are continuous, and we have

$$R_{m,n} = d^{-n}\left(\left[0, \frac{1}{m}\right[\right).$$

This shows that $R_{m,n}$ is open, hence R_m is open, so we conclude that R is a G_δ set. But R is precisely the set of points $x \in X$ that, for every neighborhood $\mathrm{B}(x; \frac{1}{m})$, there exists an integer $n \geq m$ such that $f^n(x) \in \mathrm{B}(x; \frac{1}{m})$, which is the recurrent set. ■

Proposition 3.10 (Erdős-Stone)**.** *Let* $\boldsymbol{X}$ *be a topological space and* $f : X \to X$ *a continuous dynamics. For every* $n \in \mathbb{Z}_{\geq 0}$*,*

$$\mathrm{R}(f) = \mathrm{R}(f^n).$$

3.3 Non-wandering points

Definition 3.5. Let $\boldsymbol{X}$ be a topological space and $f : X \to X$ a continuous dynamics. A *non-wandering* point of f is a point $x \in X$ such that, for every neighborhood V of x, there exists $n \in \mathbb{Z}_{>0}$ such that $f^n(V) \cap V \neq \emptyset$. The *non-wandering* set of f is the set of all non-wandering points of f, denoted $\Omega(f)$.

Proposition 3.11. *Let* $\boldsymbol{X}$ *be a topological space and* $f : X \to X$ *a continuous dynamics. Every topologically transitive point is recurrent, and every recurrent point is a non-wandering point:*

$$\mathrm{T}(f) \subseteq \mathrm{R}(f) \subseteq \Omega(f).$$

Check [MSE22] for an example of a point that is non-wandering, but is not recurrent.

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