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Metric Entropy and Folding Entropy of a Bilateral Shift with Two Alphabets

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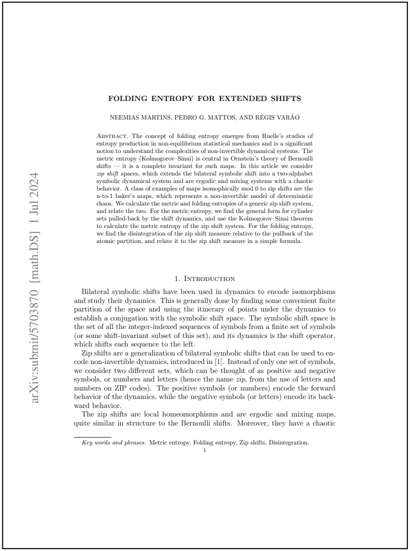

FOLDING ENTROPY FOR EXTENDED SHIFTS

NEEMIAS MARTINS, PEDRO G. MATTOS, AND RÉGIS VARÃO

ABSTRACT. The concept of folding entropy emerges from Ruelle's studies of entropy production in non-equilibrium statistical mechanics and is a significant notion to understand the complexities of non-invertible dynamical systems. The metric entropy (Kolmogorov–Sinai) is central in Ornstein's theory of Bernoulli shifts — it is a complete invariant for such maps. In this article we consider *zip shift* spaces, which extends the bilateral symbolic shift into a two-alphabet symbolic dynamical system and are ergodic and mixing systems with a chaotic behavior. A class of examples of maps isomophically mod 0 to zip shifts are the n-to-1 baker's maps, which represents a non-invertible model of deterministic chaos. We calculate the metric and folding entropies of a generic zip shift system, and relate the two. For the metric entropy, we find the general form for cylinder sets pulled-back by the shift dynamics, and use the Kolmogorov–Sinai theorem to calculate the metric entropy of the zip shift system. For the folding entropy, we find the disintegration of the zip shift measure relative to the pullback of the atomic partition, and relate it to the zip shift measure in a simple formula.

1. INTRODUCTION

Bilateral symbolic shifts have been used in dynamics to encode isomorphisms and study their dynamics. This is generally done by finding some convenient finite partition of the space and using the itinerary of points under the dynamics to establish a conjugation with the symbolic shift space. The symbolic shift space is the set of all the integer-indexed sequences of symbols from a finite set of symbols (or some shift-invariant subset of this set), and its dynamics is the shift operator, which shifts each sequence to the left.

Zip shifts are a generalization of bilateral symbolic shifts that can be used to encode non-invertible dynamics, introduced in [1]. Instead of only one set of symbols, we consider two different sets, which can be thought of as positive and negative symbols, or numbers and letters (hence the name *zip*, from the use of letters and numbers on ZIP codes). The positive symbols (or numbers) encode the forward behavior of the dynamics, while the negative symbols (or letters) encode its backward behavior.

The zip shifts are local homeomorphisms and are ergodic and mixing maps, quite similar in structure to the Bernoulli shifts. Moreover, they have a chaotic

Key words and phrases. Metric entropy, Folding entropy, Zip shifts, Disintegration.

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Baker's transformation

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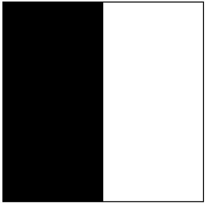
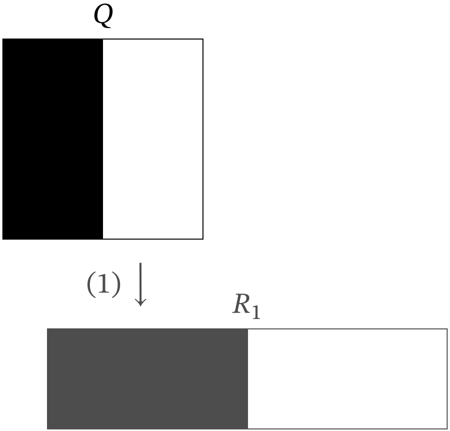


Baker's transformation



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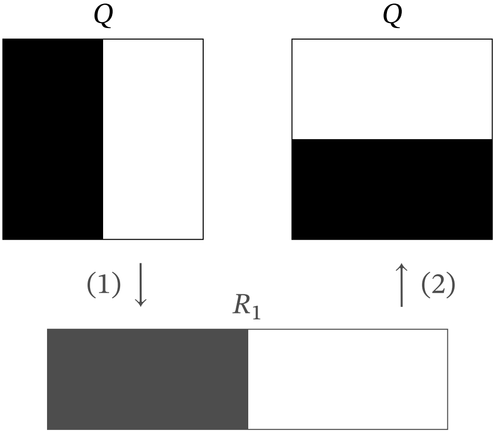

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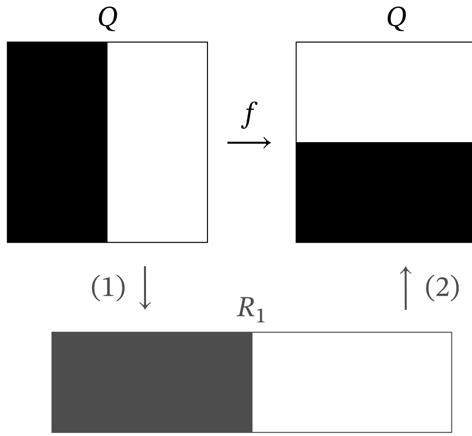
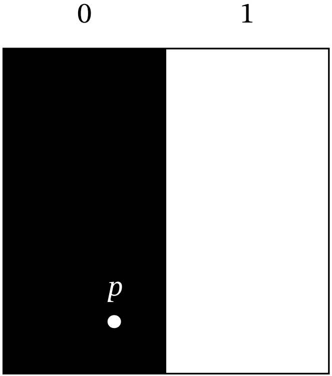
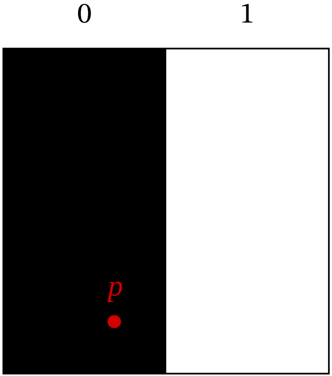


Symbolic encoding



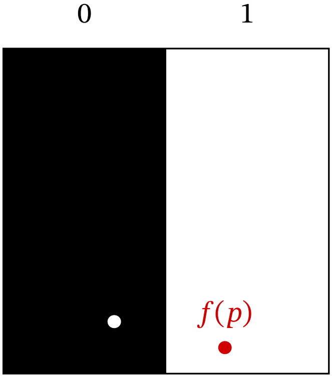
$$(\ldots, x_{-1}; x_0, x_1, x_2, \ldots)$$

Symbolic encoding



$$(\dots, x_{-1}; 0, x_1, x_2, \dots)$$

Symbolic encoding



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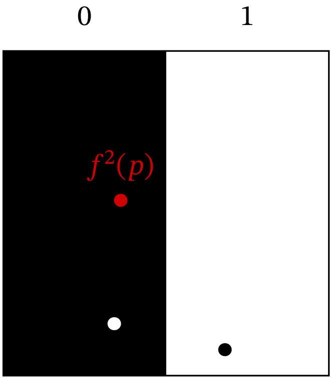
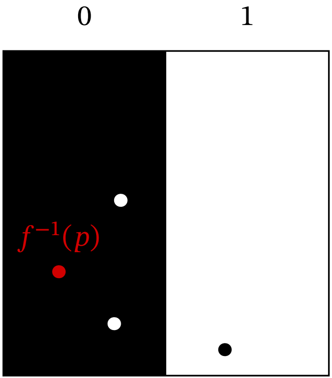
Symbolic encoding



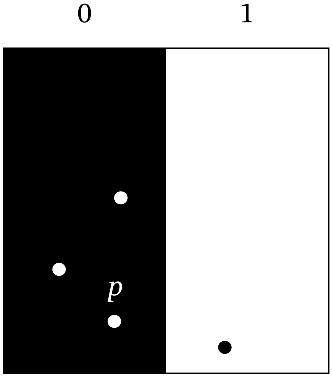
$$(\ldots, x_{-1}; 0, 1, 0, \ldots)$$

Symbolic encoding



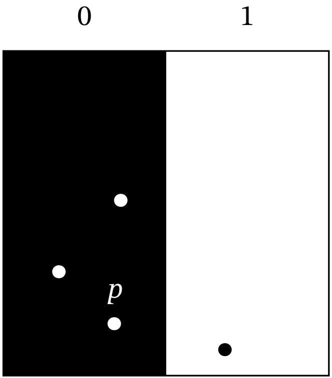
$$(\ldots, 0; 0, 1, 0, \ldots)$$

Symbolic encoding



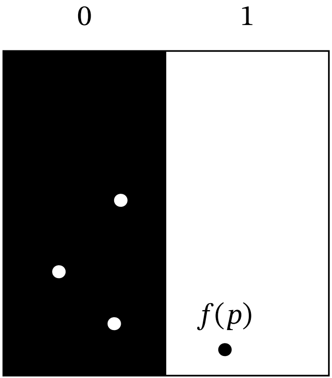
$$(\ldots, 0; 0, 1, 0, \ldots)$$

Symbolic shift



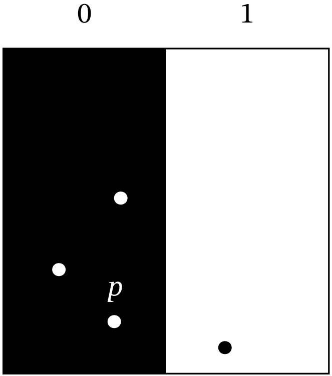
$$p \mapsto (\dots, 0; 0, 1, 0, \dots)$$

Symbolic shift



$$f(p) \mapsto (\dots, 0, 0; 1, 0, \dots)$$

Symbolic shift



$$(\ldots, x_{-2}, 0; 0, 1, 0, \ldots) \overset{\sigma}{\mapsto} (\ldots, 0, 0; 1, 0, x_3 \ldots)$$

Symbolic dynamics

Definition (Symbolic space)

$$\Sigma_n := \{(\ldots, x_{-1}; x_0, x_1, \ldots) \mid 0 \leq x_i < n\}$$

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Definition (Symbolic space)

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Definition (Symbolic shift)

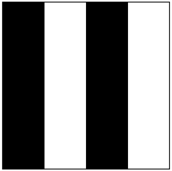
$$\begin{aligned} \sigma : \Sigma_n &\longrightarrow \Sigma_n \\ (\ldots, x_{-1}; x_0, x_1, \ldots) &\longmapsto (\ldots, x_0; x_1, x_2, \ldots) \end{aligned}$$

Section 2

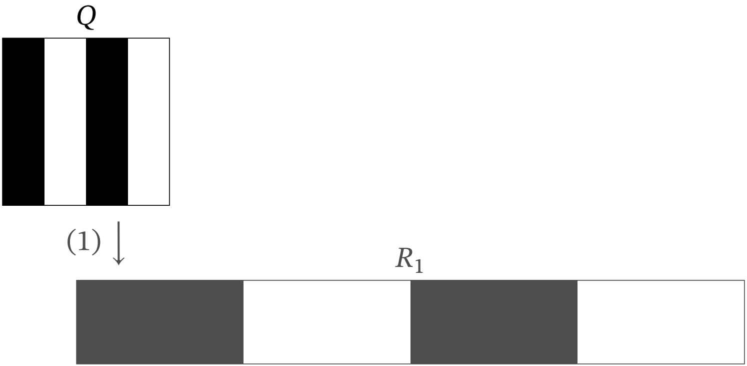
Bilateral Shifts with Two Alphabets

Baker's 2-to-1 transformation

Q



Baker's 2-to-1 transformation



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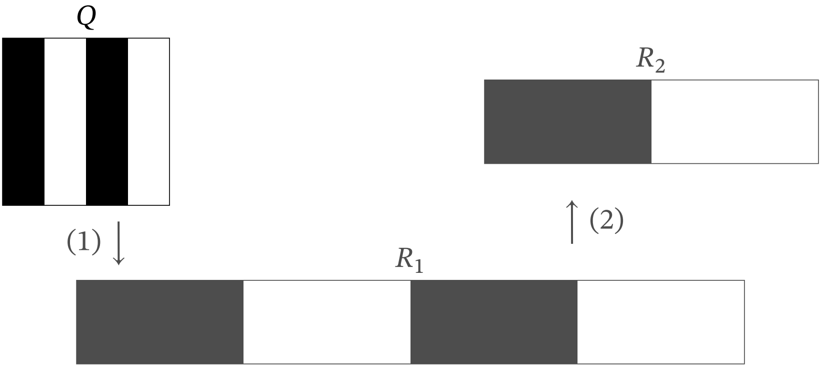

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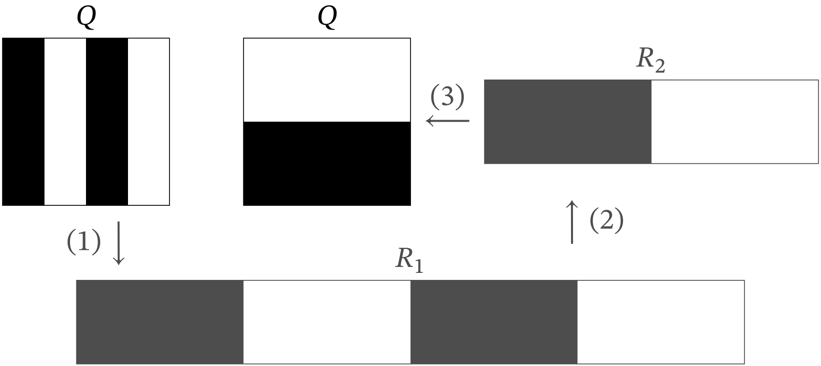
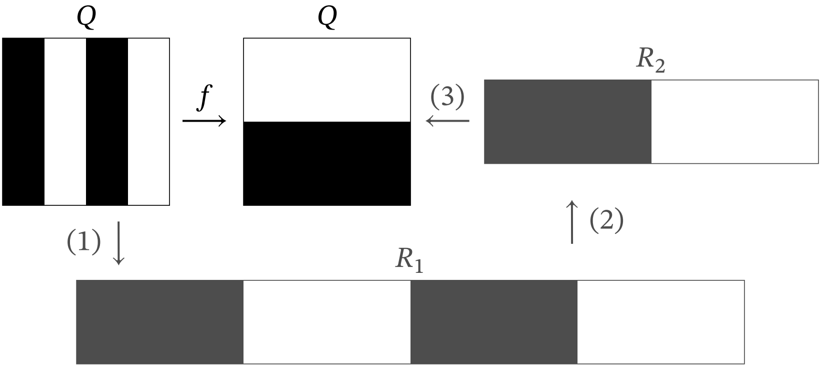
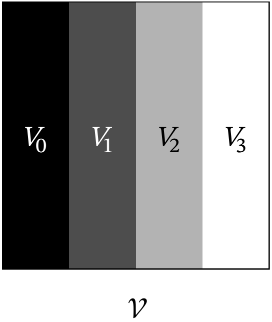
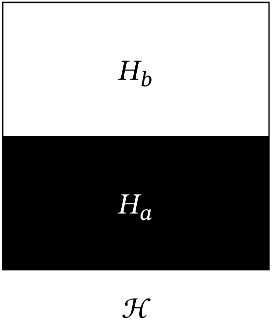


Baker's 2-to-1 transformation

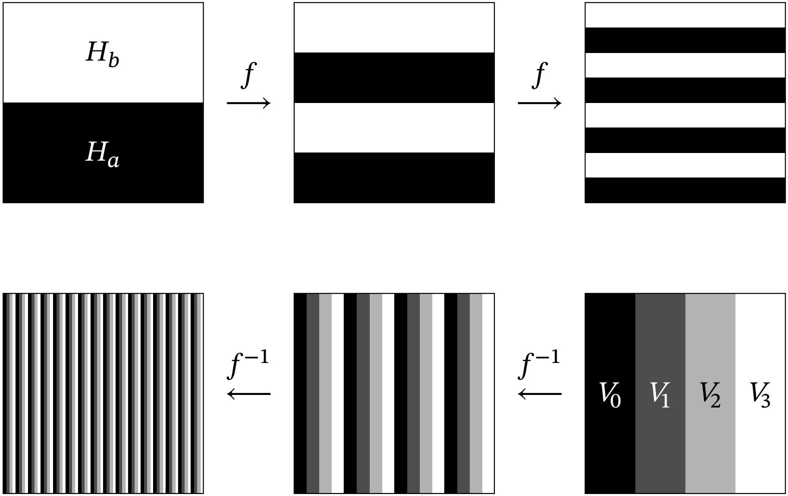


Horizontal and vertical partitions

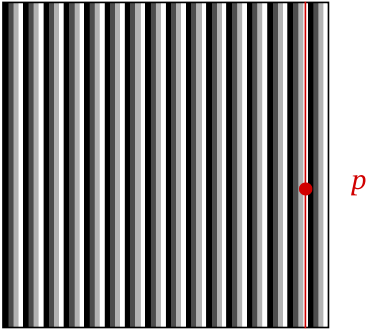




Iterated partitions



Vertical encoding



$$p \in V_{x_0} \cap f^{-1}(V_{x_1}) \cap f^{-2}(V_{x_2}) \cdots$$

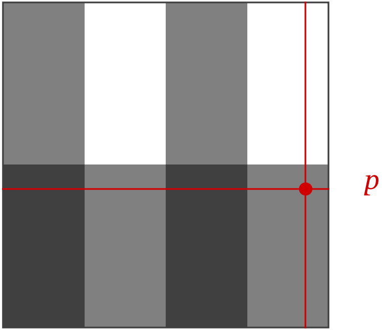
$$(x_0, x_1, x_2, \dots) = (3, 2, 0, \dots)$$

Horizontal encoding

$$p \in H_{x_{-1}} \cap f(H_{x_{-2}}) \cap f^2(H_{x_{-3}}) \cdots$$

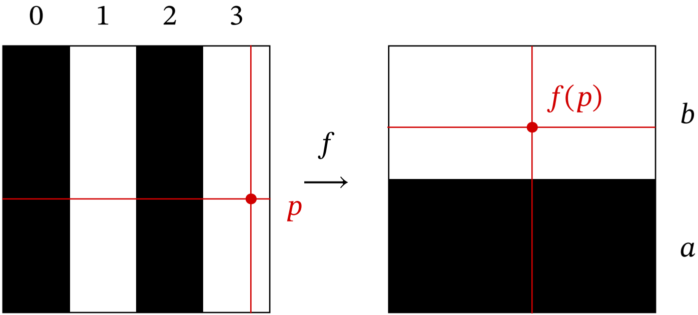
$$(\dots, x_{-3}, x_{-2}, x_{-1}) = (\dots, a, b, a)$$

Encoding



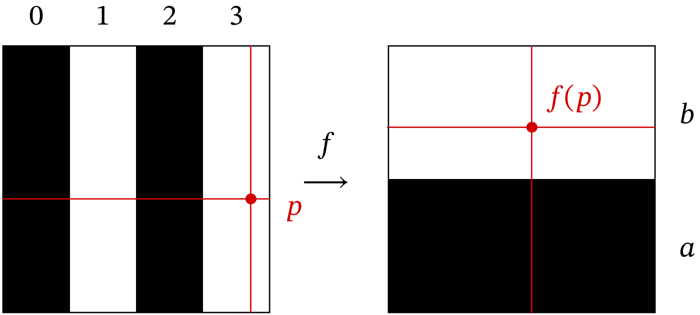
$$x = (\dots, a, b, a; 3, 2, 0, \dots)$$

Shift



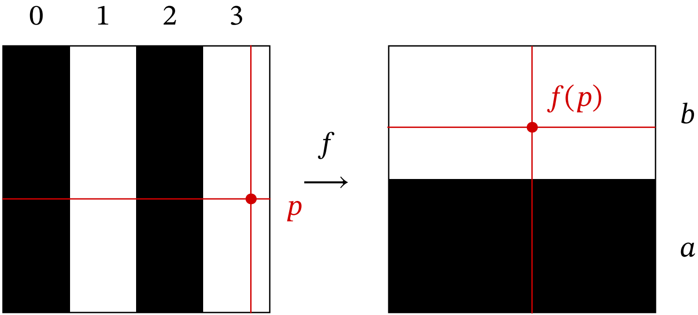
$$\phi(0) = \phi(2) = a$$
$$\phi(1) = \phi(3) = b$$

Shift



$$(\ldots, b, a; 3, 2, 0, \ldots) \overset{\sigma}{\mapsto} (\ldots, b, a, \phi(3); 2, 0, \ldots)$$

Shift



$$(\ldots, b, a; 3, 2, 0, \ldots) \overset{\sigma}{\mapsto} (\ldots, b, a, b; 2, 0, \ldots)$$

Bisymbolic dynamics

Definition (Bisymbolic space)

$$\Sigma_{n_-,n_+} := \left\{ (\dots, x_{-1}; x_0, x_1, \dots) \;\middle|\; \begin{array}{ll} 0 \le x_i < n_- & i < 0 \\ 0 \le x_i < n_+ & i \ge 0 \end{array} \right\}$$

Bisymbolic dynamics

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$$\Sigma_{n_-,n_+} := \left\{ (\dots, x_{-1}; x_0, x_1, \dots) \;\middle|\; \begin{matrix} 0 \le x_i < n_- & i < 0 \\ 0 \le x_i < n_+ & i \ge 0 \end{matrix} \right\}$$

Definition (Bisymbolic shift)

$$\phi : \{0, \dots, n_+ - 1\} \to \{0, \dots, n_- - 1\}$$

$$\begin{aligned} \sigma_\phi : \Sigma_{n_-,n_+} &\longrightarrow \Sigma_{n_-,n_+} \\ (\dots, x_{-2}, x_{-1}; x_0, x_1, \dots) &\longmapsto (\dots, x_{-1}, \phi(x_0); x_1, x_2, \dots) \end{aligned}$$

Measureable structure on Σ_{n_-,n_+}

Definition (Cylinder)

Given $i \in \mathbb{Z}$, and $s \in \{0, \dots, n_- - 1\}$ (if $i < 0$) or $s \in \{0, \dots, n_+ - 1\}$ (if $i \geq 0$), we define

$$C_i^s := \{x \in \Sigma_{n_-,n_+} \mid x_i = s\}.$$

Definition (Cylinder partitions)

$$\mathcal{C}_i := \begin{cases} \{C_i^s \mid s \in \{0, \dots, n_+ - 1\}\} & i \geq 0 \\ \{C_i^s \mid s \in \{0, \dots, n_- - 1\}\} & i < 0. \end{cases}$$

$$\mathcal{C}_{i,\dots,i'} := \bigvee_{i \leq j \leq i'} \mathcal{C}_j.$$

Measureable structure on Σ_{n_-,n_+}

Lemma (Martins, —, Varão [2])

For every $i \geq 0$,

1. $\sigma^i(\mathcal{C}_0) = \mathcal{C}_{-i}$;
2. $\sigma^{-i}(\mathcal{C}_0) = \mathcal{C}_i$;
3. $\sigma^{-i}(\mathcal{C}_{-(i+1)}) = \mathcal{C}_{-1}$;
4. $\mathcal{C}_0^n = \mathcal{C}_{0,\dots,n-1}$.
5. $\mathcal{C}_0^{\pm n} = \mathcal{C}_{-n,\dots,n-1}$.

Measure structure on Σ_{n_-,n_+}

Definition (Measure)

Probability distribution

$$p^+ = (p_0^+, \dots, p_{n_+-1}^+) \in \mathbb{\Delta}^{n_+-1}.$$

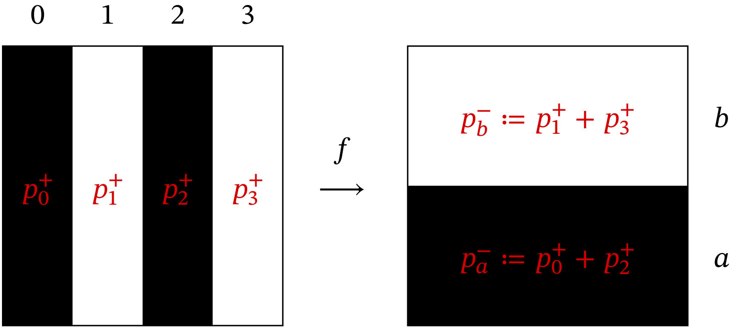
Pushforward by ϕ to $p^- \in \mathbb{\Delta}^{n_--1}$:

$$p_s^- := \sum_{r \in \phi^{-1}(s)} p_r^+.$$

Define the measure m on Σ_{n_-,n_+} by

$$m(C_i^s) := \begin{cases} p_s^- & i < 0 \\ p_s^+ & i \geq 0 \end{cases}$$

Measure structure on Σ_{n_-,n_+}



Section 3

Measure Entropy

Shannon entropy

The *(Shannon) entropy* of a distribution $p \in \mathbb{\Delta}^{d-1}$ is

$$\mathrm{H}_{\mathbb{\Delta}}(p) \coloneqq \sum_{i=0}^{d-1} -p_i \log(p_i).$$

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Let $w \in \mathbb{\Delta}^{n-1}$. The *w-weighted sum* of n distributions $q^i \in \mathbb{\Delta}^{k_i - 1}$

$$\bigoplus_{i=0}^{n-1} w_i q^i := (w_0 q_0^0, \dots, w_0 q_{k_0-1}^0, \cdots, w_{n-1} q_0^{n-1}, \dots, w_{n-1} q_{k_{n-1}-1}^{n-1}).$$

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$$\frac{1}{3}\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right) \oplus \frac{2}{3}(1,0) = \left(\frac{1}{6}, \frac{1}{12}, \frac{1}{12}, \frac{2}{3}, 0\right)$$

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$\mathrm{H}_{\mathbb{\Delta}}$ satisfies the *weighted additivity* property

$$\mathrm{H}_{\mathbb{\Delta}}\Big(\bigoplus_{i\in[n]} w_i q^i\Big) = \mathrm{H}_{\mathbb{\Delta}}(w) + \sum_{i=0}^{n-1} w_i \mathrm{H}_{\mathbb{\Delta}}(q^i).$$

Measure entropy

The *entropy* of a partition $\mathcal{P}$ relative to a measure m is

$$\mathrm{H}_m(\mathcal{P}) \coloneqq \sum_{P \in \mathcal{P}} -m(P) \log(m(P)).$$

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The *entropy* of a dynamics $f\colon X \to X$ relative to $\mathcal{P}$ is

$$\mathrm{h}_m(f, \mathcal{P}) := \lim_{n \to \infty} \frac{1}{n} \mathrm{H}_m(\mathcal{P}^n).$$

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The *entropy* of a dynamics $f\colon X \to X$ is

$$\mathrm{h}_m(f) \coloneqq \sup_{\mathcal{P}} \mathrm{h}_m(f, \mathcal{P}).$$

Entropy of partition by cylinders

Lemma (Martins, M—, Varão [2])

1. $\mathrm{H}_m(\mathcal{C}_0) = \mathrm{H}_{\triangle}(p^+)$;
2. $\mathrm{H}_m(\mathcal{C}_{-1}) = \mathrm{H}_{\triangle}(p^-)$;
3. $\mathrm{H}_m(\mathcal{C}_{-n,\dots,0,\dots,n'-1}) = n\mathrm{H}_m(\mathcal{C}_{-1}) + n'\mathrm{H}_m(\mathcal{C}_0)$.

Kolmogorov–Sinai theorem

Theorem ([5, Theorem 9.2.1])

Let

- $f : X \to X$ *be a measure-preserving transformation;*
- $(\mathcal{P}_k)_{k\in\mathbb{N}}$ *be an increasing sequence of partitions generating* $\mathcal{M}$.

Then

$$\mathrm{h}_m(f) = \lim_{n\to\infty} \mathrm{h}_m(f, \mathcal{P}_n).$$

Increasing sequence of partitions

Definition

$\mathcal{P}_n := \mathcal{C}_0^{\pm n} = \mathcal{C}_{-n,\dots,n-1}$.

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We want to calculate

$$\mathrm{h}_m(\sigma, \mathcal{P}_n) = \lim_{k\to\infty} \frac{1}{k} \mathrm{H}_m(\mathcal{P}_n^k),$$

so we need to calculate $\mathcal{P}_n^k$.

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so we need to calculate $\mathcal{P}_n^k$.

Lemma (Martins, M—, Varão [2])

$\mathcal{P}_n^k = \mathcal{C}_{-n,\ldots,n+k-2}$ *(for $n \geq 1$, $k \geq 2n$).*

Complicated proof.

Measure entropy

Lemma (Martins, M—, Varão [2])

$$\mathrm{h}_m(\sigma, \mathcal{P}_n) = \mathrm{H}_m(\mathcal{C}_0).$$

Measure entropy

Lemma (Martins, M—, Varão [2])

$$h_m(\sigma, \mathcal{P}_n) = H_m(\mathcal{C}_0).$$

Proof.

$$H_m(\mathcal{P}_n^k) = H_m(\mathcal{C}_{-n,\ldots,n-1+k-1}) = nH_m(\mathcal{C}_{-1}) + (n+k-1)H_m(\mathcal{C}_0),$$

Measure entropy

Lemma (Martins, M—, Varão [2])

$$\mathrm{h}_m(\sigma, \mathcal{P}_n) = \mathrm{H}_m(\mathcal{C}_0).$$

Proof.

$$\mathrm{H}_m(\mathcal{P}_n^k) = \mathrm{H}_m(\mathcal{C}_{-n,\ldots,n-1+k-1}) = n\mathrm{H}_m(\mathcal{C}_{-1}) + (n+k-1)\mathrm{H}_m(\mathcal{C}_0),$$

therefore

$$\begin{aligned}\mathrm{h}_m(\sigma, \mathcal{P}_n) &= \lim_{k\to\infty} \frac{1}{k}\mathrm{H}_m(\mathcal{P}_n^k) \\ &= \mathrm{H}_m(\mathcal{C}_0).\end{aligned}$$

□

Measure entropy

Theorem (Martins, M—, Varão [2])

$$\mathrm{h}_m(\sigma) = \mathrm{H}_m(\mathcal{C}_0).$$

Measure entropy

Theorem (Martins, M—, Varão [2])

$$h_m(\sigma) = H_m(\mathcal{C}_0).$$

Proof.

By Kolmogorov-Sinai and the lemmas,

$$\begin{aligned} h_m(\sigma) &= \lim_{n\to\infty} h_m(\sigma, \mathcal{P}_n) \\ &= \lim_{n\to\infty} H_m(\mathcal{C}_0) \\ &= H_m(\mathcal{C}_0). \end{aligned}$$

□

Section 4

Folding Entropy

Conditional entropy

Defined using the disintegration $(m_{P'})_{P' \in \mathcal{P}'}$ of m with respect to $\mathcal{P}'$.

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Definition (Rokhlin [3])

The *conditional entropy of $\mathcal{P}$ with respect to $\mathcal{P}'$* is

$$\mathrm{H}_m(\mathcal{P} \mid \mathcal{P}') := \int_{P' \in \mathcal{P}'} \mathrm{H}_{m_{P'}}(\mathcal{P}|_{P'}) m_{\mathcal{P}'}(\mathrm{d}P').$$

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$$\mathrm{H}_m(\mathcal{P} \mid \mathcal{P}') := \int_{P' \in \mathcal{P}'} \mathrm{H}_{m_{P'}}(\mathcal{P}|_{P'}) m_{\mathcal{P}'}(\mathrm{d}P').$$

For countable $\mathcal{P}, \mathcal{P}'$

$$\mathrm{H}_m(\mathcal{P} \mid \mathcal{P}') = \sum_{P \in \mathcal{P}} \sum_{P' \in \mathcal{P}'} -m(P \cap P') \log\left(\frac{m(P \cap P')}{m(P')}\right).$$

Folding entropy

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Definition

Given the atomic partition

$$\epsilon := \{\{x\} \mid x \in X\},$$

the *folding entropy* of f with respect to m is

$$\mathcal{F}(f) := \mathrm{H}_m(\epsilon \mid f^{-1}(\epsilon)).$$

Properties of the folding entropy

- $\mathcal{F}(f) \leq \log n$ when f is n-to-1;

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Folding entropy of $(\Sigma_{n_-,n_+}, \sigma)$

In our case

$$\mathrm{H}_m(\epsilon \mid \sigma^{-1}(\epsilon)) = \int_{\hat{x}\in\sigma^{-1}(\epsilon)} \mathrm{H}_{m_{\hat{x}}}(\epsilon|_{\hat{x}})\hat{m}(\mathrm{d}\hat{x}),$$

where

- $\hat{x}(s) := (\dots, x_{-2}; s, x_0, \dots);$

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Disintegration

For each $r \in \{0, \dots, n_- - 1\}$, we define a distribution q^r on $\phi^{-1}(r)$ by

$$q^r_s := \frac{p^+_s}{p^-_r} \qquad \forall s \in \phi^{-1}(r).$$

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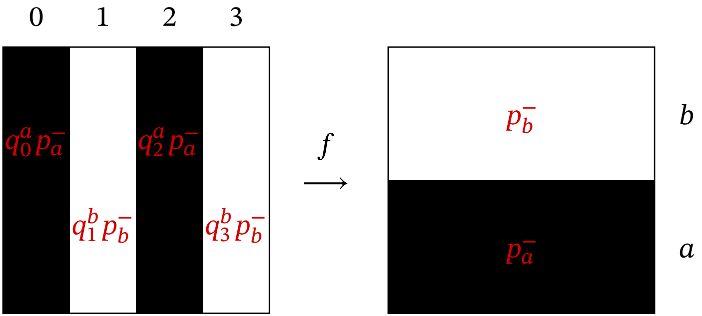

Disintegration

Proposition (Martins, —, Varão [2])

The probability $m_{\hat{x}}$ defined on $\hat{x}$ by

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Long but straighforward calculations.

Folding entropy

Theorem (Martins, M—, Varão [2])

$$\mathcal{F}(\sigma) = \sum_{0 \leq s < n_-} \mathrm{H}_{\Delta}(q^s) p_s^- = \mathrm{H}_m(\mathcal{C}_0) - \mathrm{H}_m(\mathcal{C}_{-1}).$$

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Proof.

$$\begin{aligned}
\mathcal{F}(\sigma) &= \mathrm{H}_m(\epsilon \mid \sigma^{-1}(\epsilon)) \\
&= \int_{\hat{x}\in\sigma^{-1}(\epsilon)} \mathrm{H}_{m_{\hat{x}}}(\epsilon|_{\hat{x}})\hat{m}(\mathrm{d}\hat{x}) \\
&= \sum_{0\le s<n_-} \int_{\hat{x}\in\hat{C}^s_{-1}} \mathrm{H}_{m_{\hat{x}}}(\epsilon|_{\hat{x}})\hat{m}(\mathrm{d}\hat{x}) \\
&= \sum_{0\le s<n_-} \mathrm{H}_{\Delta}(q^s)p_s^-. \qquad \square
\end{aligned}$$

Folding entropy

The relation

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hence

$$\mathrm{H}_{\Delta}(p^+) = \mathrm{H}_{\Delta}(p^-) + \sum_{0 \le s < n_-} \mathrm{H}_{\Delta}(q^s) p_s^- .$$

Relation of measure entropy and folding entropy

Since $\mathrm{h}_m(\sigma) = \mathrm{H}_m(\mathcal{C}_0)$, if follows that

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Section 5

Questions

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2. In particular, is it conjugate to some standard unilateral shift?
3. What are other examples of endomorphisms which can be conjugated with $(\Sigma_{n_-,n_+}, \sigma)$?

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