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Entropy and Non-Deterministic Chaos for Piecewise Smooth Vector Fields

MAT80 - IMECC
28 November, 2024

O. M. L. Gomide, P. G. Mattos, and R. Varão. “The Orbit Space Approach for Piecewise Smooth Vector Fields”. 2023 [7]

arXiv:2310.13804v1 [math.DS] 20 Oct 2023

THE ORBIT SPACE APPROACH FOR PIECEWISE SMOOTH VECTOR FIELDS

OTÁVIO M. L. GOMIDE, PEDRO G. MATTOS, AND RÉGIS VARÃO

Abstract. In this work we establish a well defined theory of ‘orbit spaces’ for nonsmooth vector fields (NSVF). This approach is inspired from the techniques already used in the study of endomorphism, namely inverse limit analysis. We then apply the construction of our theory for the understanding of transitivity in NSVF. We also prove that the known examples of transitive NSVF are indeed transitive in the ‘orbit space’ as a consequence of our general theorem of transitivity.

1. Introduction

Nonsmooth vector fields (NSVF) have been extensively studied in the last years due to their applicability to model real world problems. In light of this, there is a natural interest in understanding the dynamics associated with them, which is very complicated and presents fascinating behaviors.

The novelties coming from the NSVF compared to the continuous case resides on a proper or convinient way to define a solution and its behaviour when two distinct vector fields meet (namely at the ‘discontinuity manifold’ or ‘switching manifold’). There are several ways to define the solutions of a NSVF, and each of them gives rise to a new dynamics [12, 11]; nevertheless, there is a special interest in the dynamics given by Filippov’s convention, since it provides very accurate models for different kinds of problems. The nonuniqueness of solutions is an important feature of the NSVF, which gives rise to some regions on the discontinuity manifold known as sliding and escaping regions. Many orbits of a NSVF can visit the same point, which is a behavior that does not occur for continuous vector fields.

Many works have studied the dynamics of NSVF, but our work has as one of the goals to propose a new way to understand NSVF. Due to the nature of NSVF having different orbits going through the same points we understand that it may cause a lack of understanding of the true meaning of the system as a whole. Therefore, in this work we provide a way to understand how the nonuniqueness of solutions impacts the dynamics of a NSVF, and we do it so by introducing an associated dynamical system which is able to restore the uniqueness property.

Among all properties of smooth dynamical systems, transitivity has always been one of the most fundamental ones. A transitive system is a system that has a dense orbit. This is an interesting concept by itself but it is worth mentioning that transitivity is also one of the ingredients of chaos. Despite the importance of NSVF, there is still a lack of results about its global dynamics. We may claim that this is due to the scenario of nonuniqueness of solutions, which we propose to properly overcome in the present work.

In [2, 4], the authors provided some chaotic examples of 2-dimensional NSVF displaying a dense orbit in the phase space. These two articles explore intensively the nonuniqueness of solution, in particular the escaping regions. These are regions on which many different orbits go through the same points repeatedly. Escaping regions produce so many different orbits that one may have a feeling that transitivity is forcibly more common in the NSVF scenario. This is in some sense true, but one should see it as due to the richness of the dynamics. For instance, in [4] the authors provide a first example of transitivity for NSVF on a two-sphere, which is interesting and shows the richness of NSVF, since a continuous transitive flow on the two-sphere is long known not to exist.

In this work we propose to understand the transitivity of a NSVF in a more appropriate space. This space will be called the “orbit space” which is in fact the space of all orbits. This is inspired by the technique known as “inverse limit” widely studied from those which deal with endomorphisms[3, 9, 10]. An endomorphism is a noninvertible map, hence the inverse limit is simply the space of all possible orbits and there is a very natural dynamics in this space which is just the translation of the original dynamics to this space (see §3). We apply this ideia to the NSVF since it is a noninvertible system. The orbit space then carries all the complexity of the system, but without the issues of having multiple orbits going through the same place.

Our goal is to propose the use of the orbit space in NSVF. So our first main result is in fact a series of results with the purpose of establishing a well defined theory of “orbit space” for NSVF. This is done on §3 where we define the orbit space and its dynamics, provide two topologically equivalent distance functions for this space, and provide some results that help the use of such space for the NSVF analysis. Following this orbit space section, we prove a result that establishes when a NSVF is transitive in the orbit space, the space where we

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Key words and phrases. piecewise smooth differential system, Filippov system, transitivity.

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R. D. Euzébio, P. G. Mattos, and R. Varão. "Characterization of Non-Deterministic Chaos in Two-Dimensional Non-Smooth Vector Fields". 2024 [6]


CHARACTERIZATION OF NON-DETERMINISTIC CHAOS IN TWO-DIMENSIONAL NON-SMOOTH VECTOR FIELDS

RODRIGO D. EUZÉBIO, PEDRO G. MATTOS, AND RÉGIS VARÃO

Abstract. Our context is Filippov systems defined on two-dimensional manifolds having a finite number of tangency points. We prove that topological transitivity is a necessary and sufficient condition for the occurrence of non-deterministic chaos when the Filippov system has non-empty sliding or escaping regions. A fundamental result for continuous flows is the equivalence of topological transitivity and existence of a dense orbit. We prove in our setting that topological transitivity for Filippov systems is indeed equivalent to the existence of a dense Filippov orbit, although, in contrast to the continuous case, we are not able to guarantee that the dense orbit implies the existence of a residual set of dense orbits. Finally, we prove that, in this context, topological transitivity implies strictly positive topological entropy for the Filippov system. This calculation is made using techniques similar to those from symbolic dynamics.

1. Introduction

Ordinary differential equations (ODEs) are well-established tools for modeling a wide range of real-world problems. Recently, however, the consideration of discontinuities in ODE solutions has gained prominence in the study of dynamical systems. Discontinuous ODEs appear in various applications, such as control theory, the dynamics of bouncing balls, predator foraging, anti-lock braking systems (ABS), and mechanical systems with component collisions, friction, sliding, or squealing. For detailed discussions on these applications, refer to [10], [4], [22], [11], [15], [18], [19], [20], [9] and related references.

One prominent approach to discontinuous ODEs is Filippov's theory (see [14]), which allows solutions to "slide" through discontinuity regions according to predefined rules. This method can lead to non-unique solutions and introduce unusual behaviors, even in familiar scenarios such as planar phase portraits. Alternative approaches to discontinuous ODEs are well-articulated in Cortés [7]. In this paper, we focus on ODEs with discontinuities, referred to as *Filippov systems*, in accordance with Filippov's convention.

We investigate the chaotic behavior of Filippov systems defined on two-dimensional manifolds. In the realm of smooth deterministic dynamical systems, chaos is traditionally defined by Devaney as a system that is topologically transitive, sensitive to initial conditions, and has a dense set of periodic orbits. In [2], the authors refine these conditions to show that topological transitivity and density of periodic solutions are sufficient for chaos. We extend this concept to Filippov systems, where the notion of chaos is adjusted to account for the non-deterministic nature of forward orbits. For further insights into non-deterministic chaos, see [5], [6], [12], and [21]. The definition of *non-deterministic chaos* is detailed in Section 3.

2010 *Mathematics Subject Classification.* 34A36, 34A60, 37G15, 37B40.
Key words and phrases. piecewise smooth differential system, Filippov system, transitivity, topological entropy.

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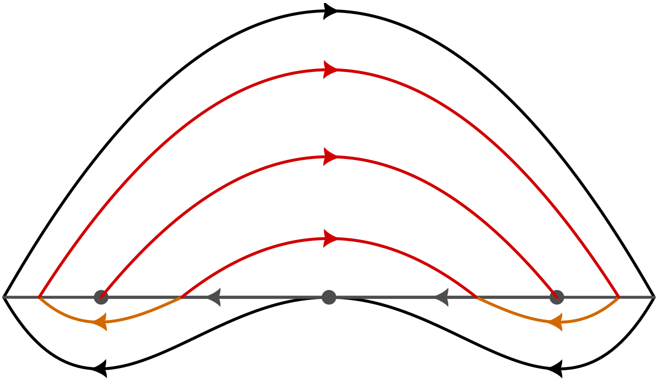
Section 1

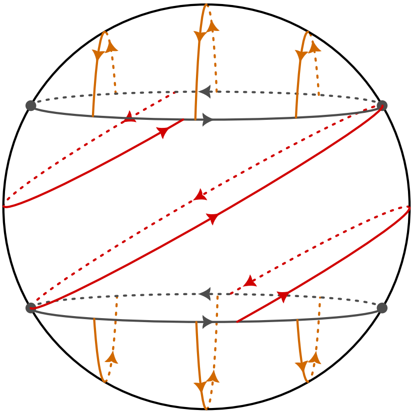
Piecewise smooth vector fields

Bean model



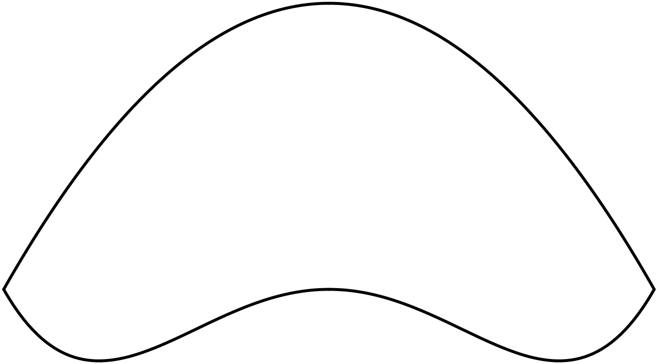
C. A. Buzzi, T. de Carvalho, and R. D. Euzébio. "Chaotic Planar Piecewise Smooth Vector Fields with Non-Trivial Minimal Sets". *Ergodic Theory and Dynamical Systems* 36.2 (2016), pp. 458–469

Sphere model

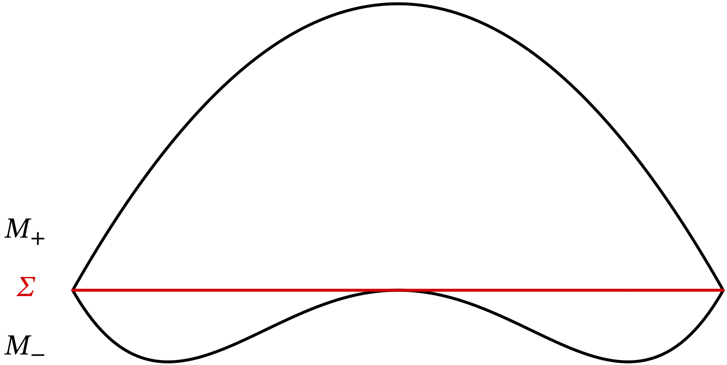


R. D. Euzébio, J. S. Jucá, and R. Varão. “There Exist Transitive Piecewise Smooth Vector Fields on $\mathbb{S}^2$ but not Robustly Transitive”. *Journal of Nonlinear Science* 32.4 (2022), No. 55

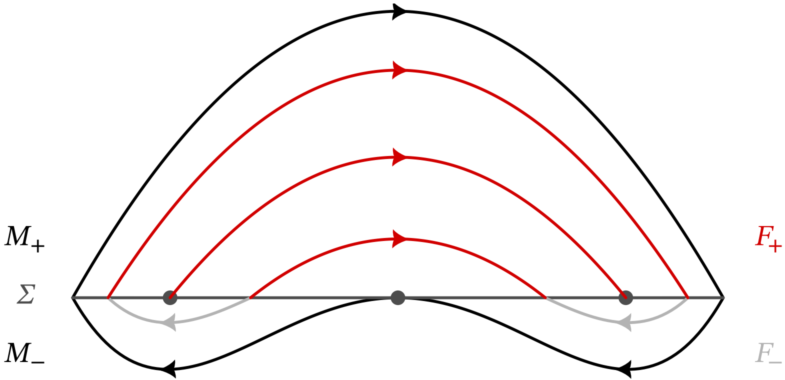
Manifold



Switching manifold



Piecewise smooth vector field



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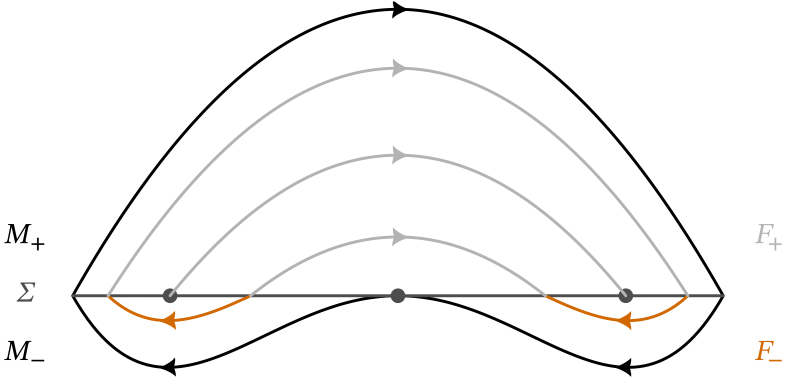
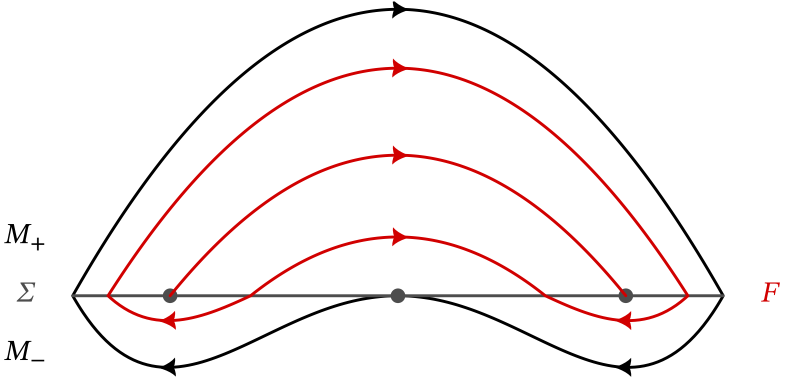


Piecewise smooth vector field



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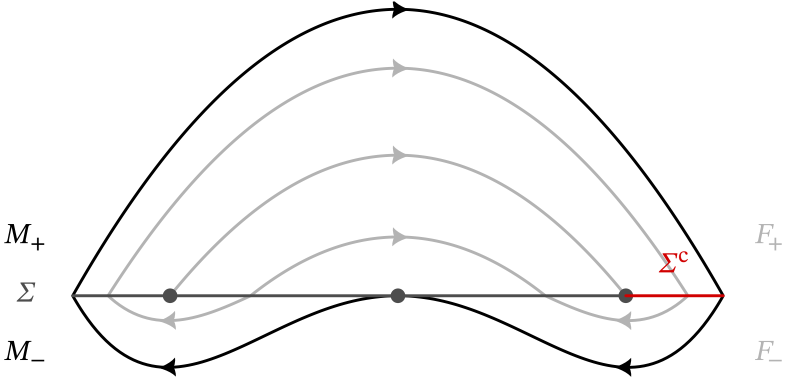

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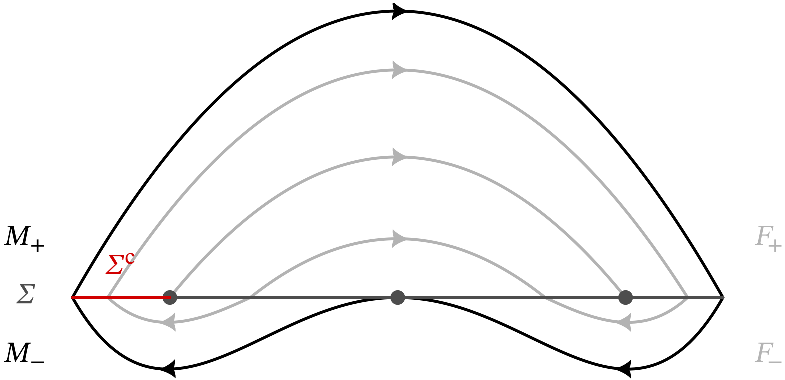
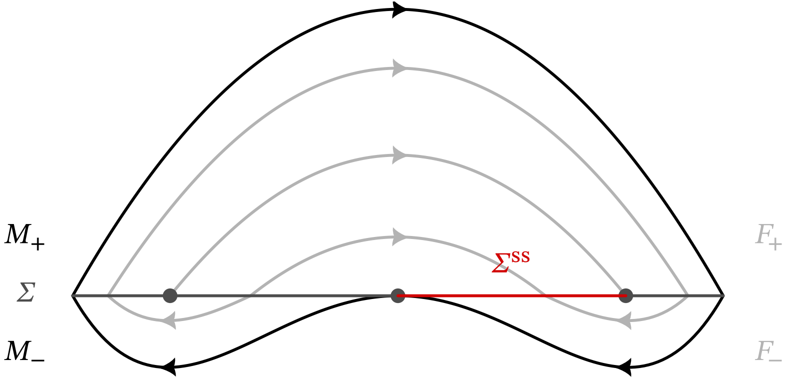
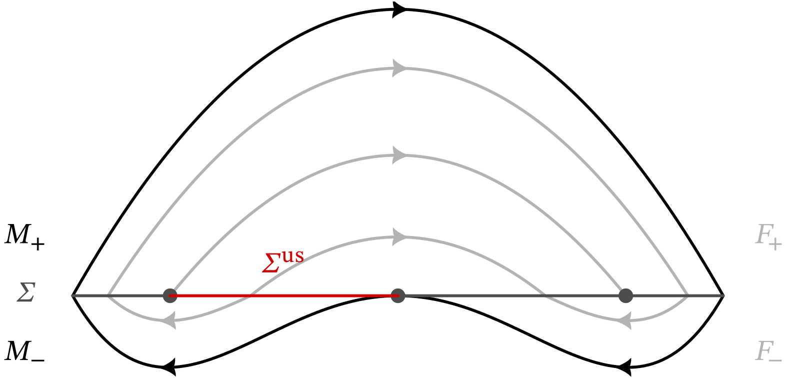


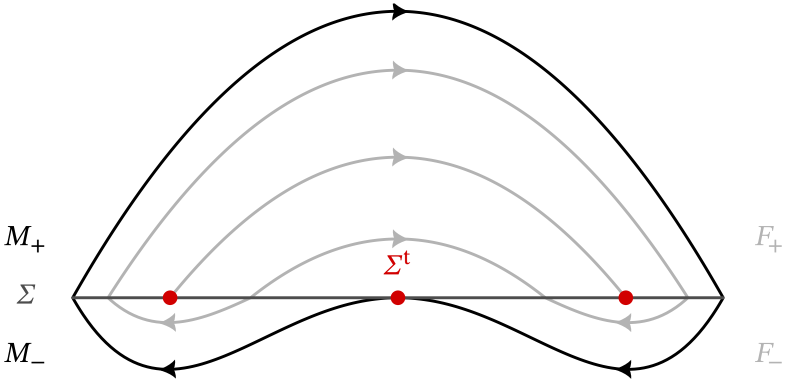
Sliding region



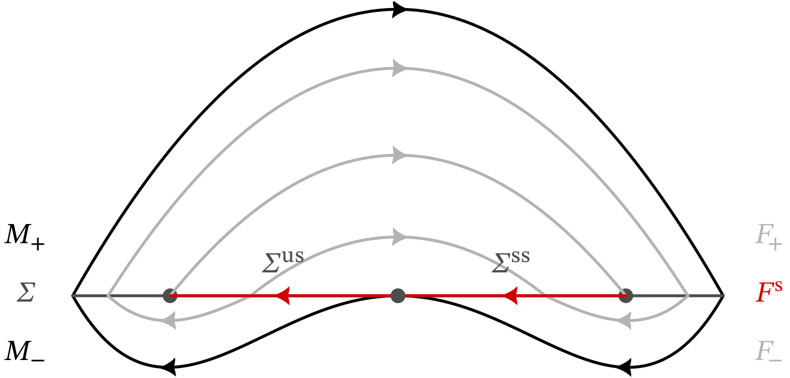
Sliding region



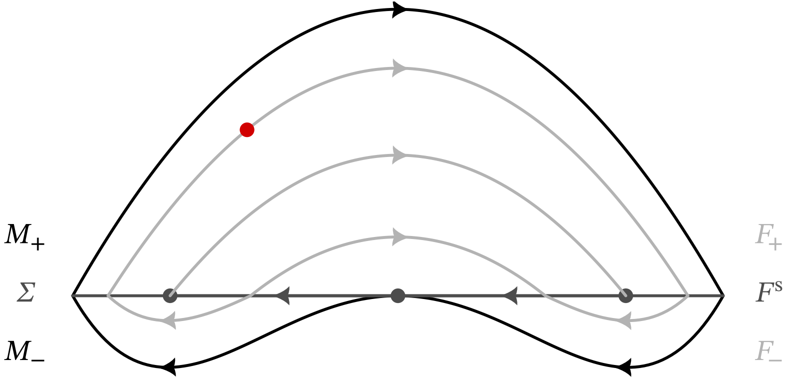
Tangent region



Sliding vector field



Orbits of a PSVF



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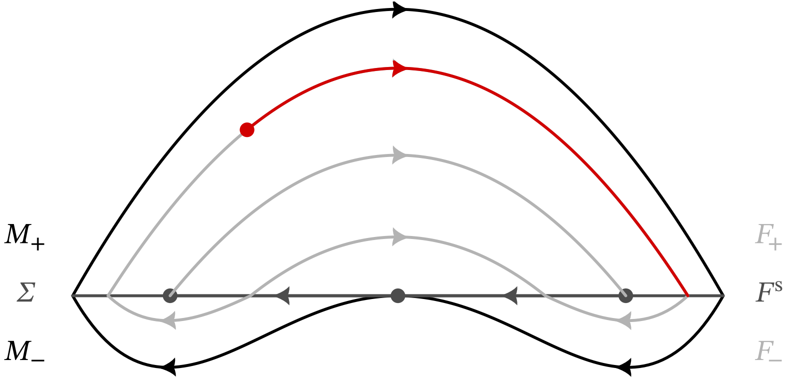

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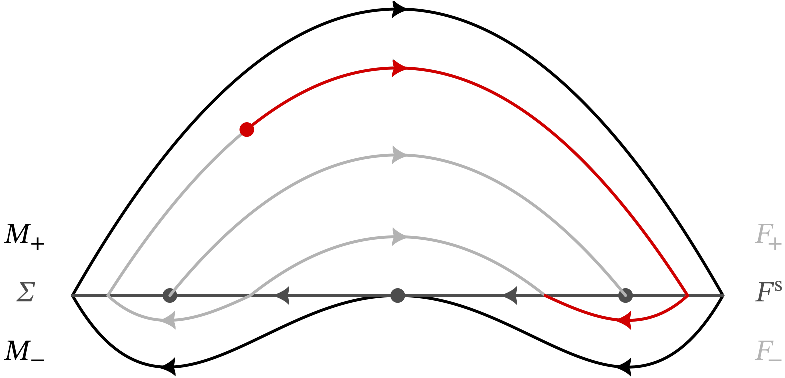

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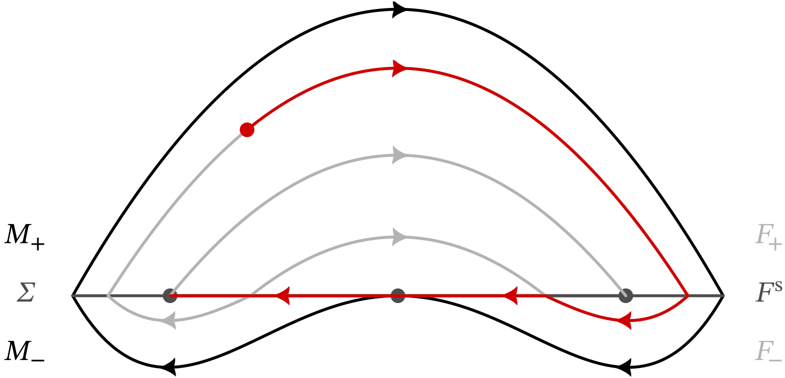

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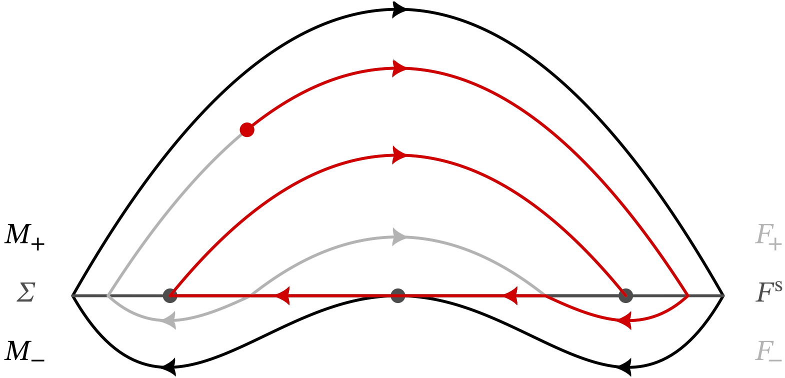

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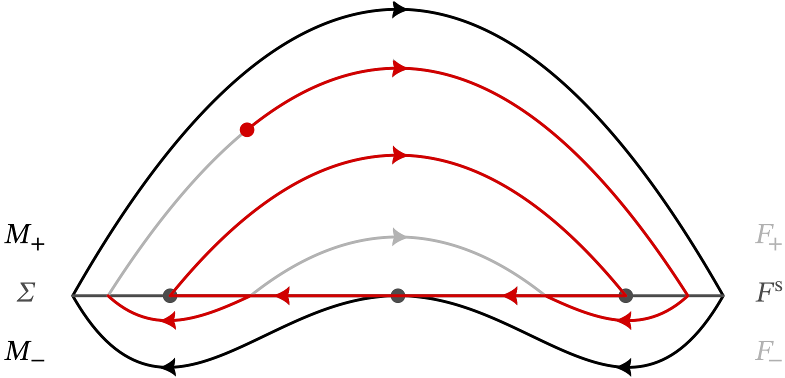

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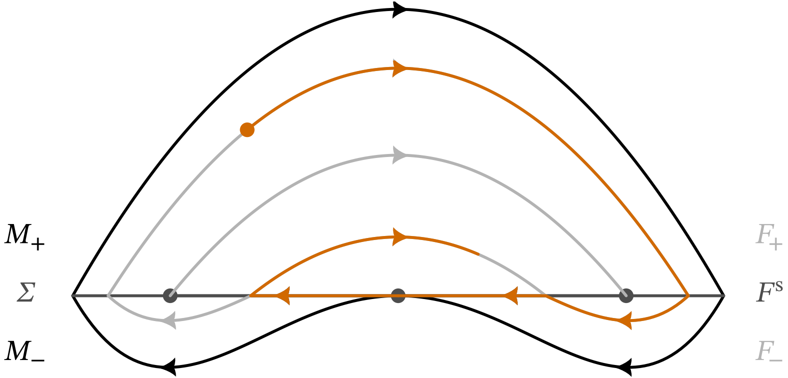
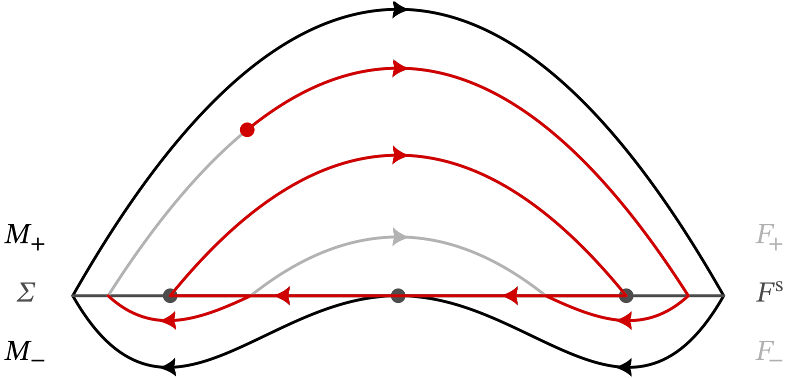


Orbits of a PSVF



Section 2

Non-deterministic chaos

Devaney chaos

Definition (Devaney [4])

A continuous map $f : X \to X$ on a metric space is *chaotic* when

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1. it is *transitive*
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Theorem (Banks et al. [2])

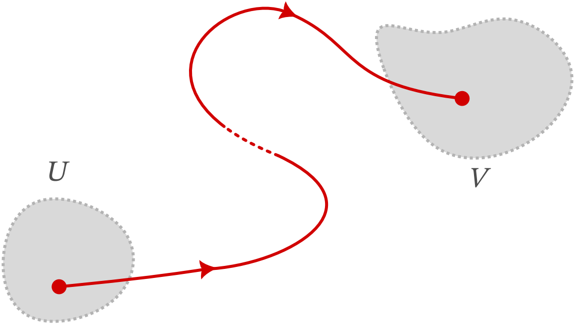
For a continuous map $f : X \to X$ on a metric space:

1 and 2 implies 3.

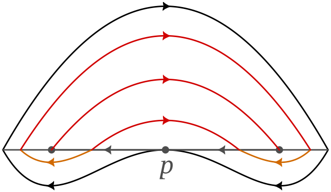
Transitivity

Definition

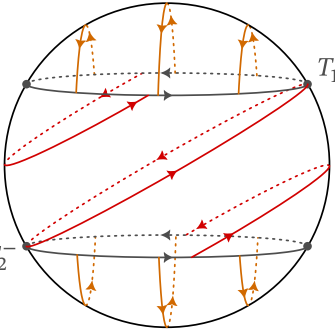
F is *topologically transitive* when, given any two non-empty open sets U and V of M, there exists a Filippov orbit from a point of U to a point of V.



Transitive examples



(1) Bean model

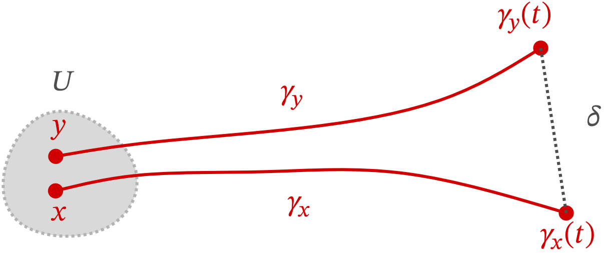


(2) Sphere model

Sensitivity

Definition

F is *sensitive (to initial conditions)* when there exists a fixed $\delta > 0$ such that, for any non-empty open set U, there exist points $x, y \in U$, Filippov orbits γ_x and γ_y which start at x and y, respectively, and some time $t > 0$ such that $d(\gamma_x(t), \gamma_y(t)) > \delta$.



Non-deterministic chaos

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2. periodic orbits are dense
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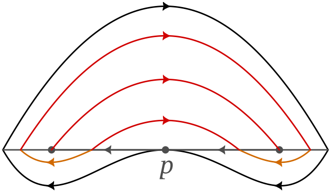
Theorem (Euzébio, M—, Varão [6])

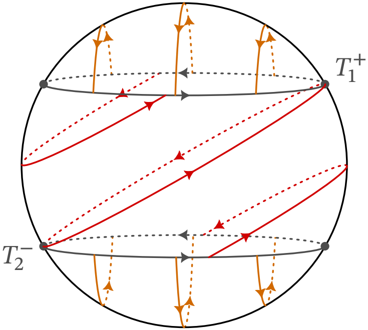
For F a 2-dimensional Filippov PSVF with $\Sigma^s \neq \emptyset$:

$$F \text{ transitive} \Rightarrow F \text{ chaotic.}$$

Artificial transitivity?



(1) Bean model



(2) Sphere model

Section 3

Orbit spaces

Why orbit spaces?

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- Disentangle orbits

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- Recover uniqueness of solutions

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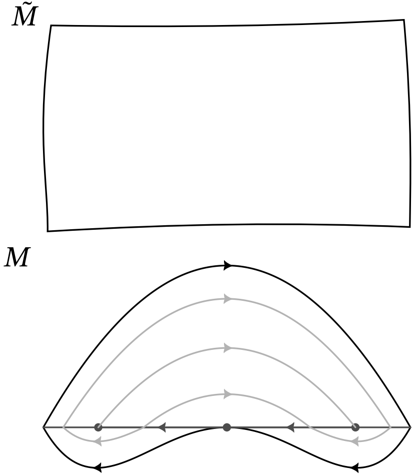
- Study global properties of the dynamics

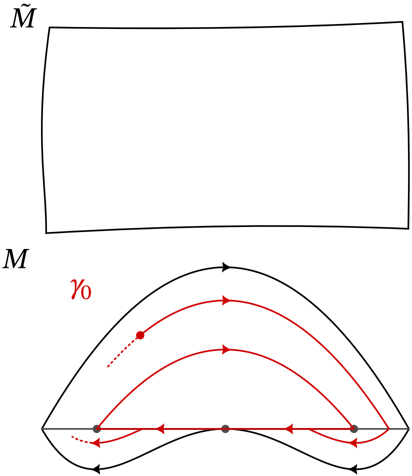
Why orbit spaces?

- Disentangle orbits
- Recover uniqueness of solutions
- Study global properties of the dynamics
- Motivation: endomorphisms

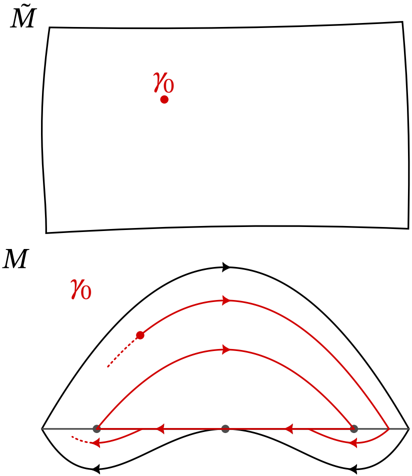
Orbit space



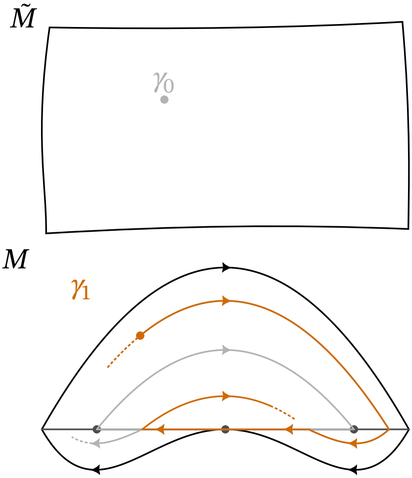
Orbit space



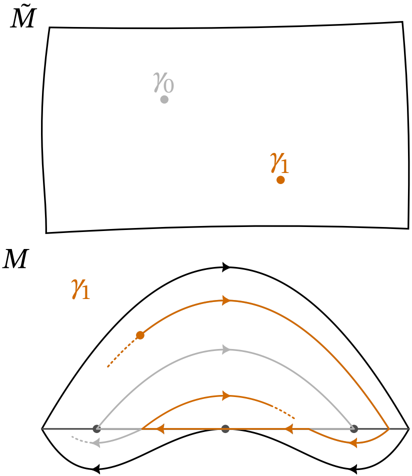
Orbit space



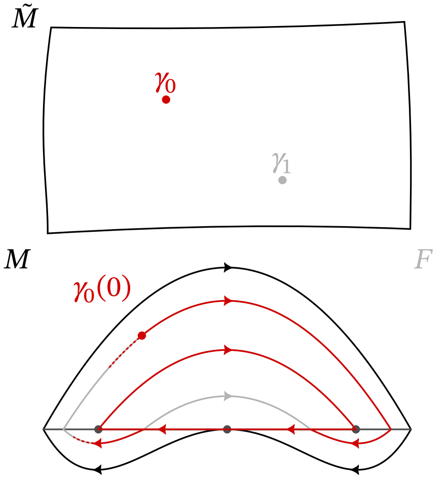
Orbit space



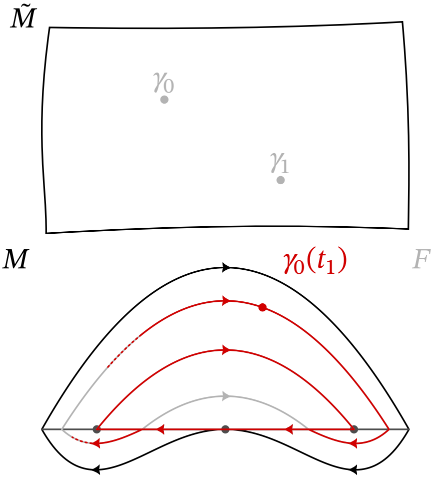
Orbit space



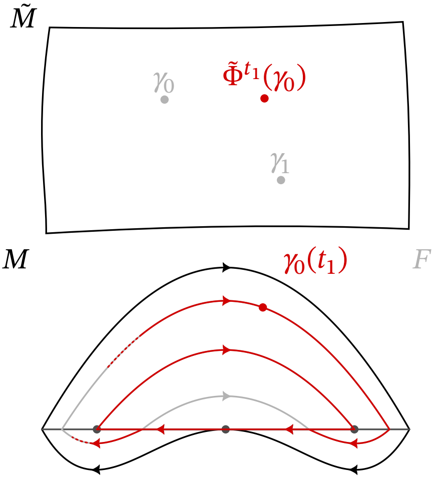
Orbit space flow



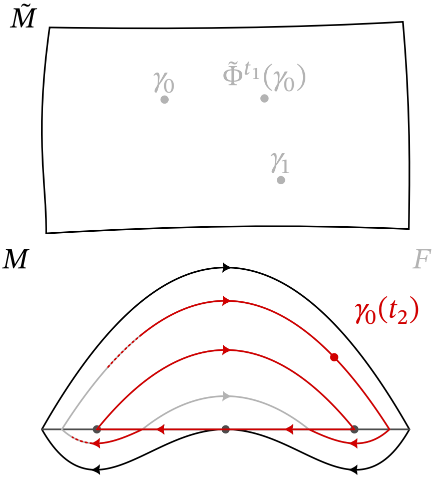
Orbit space flow



Orbit space flow



Orbit space flow



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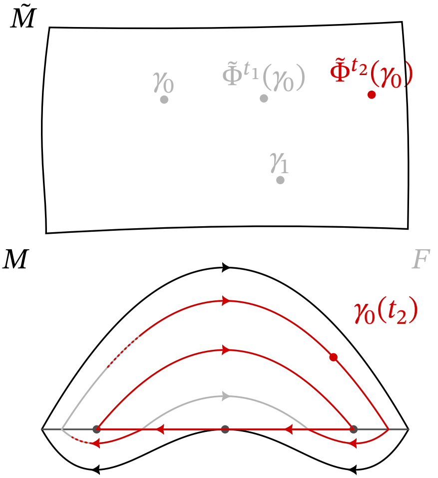
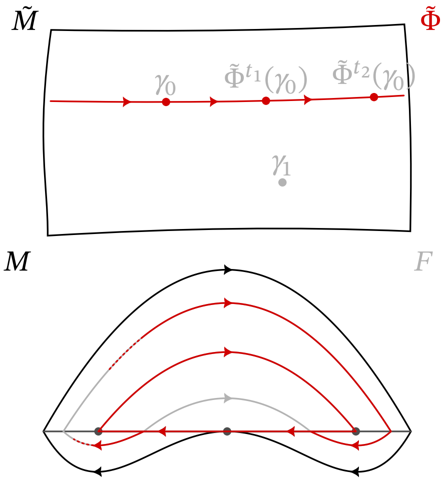


Orbit space flow



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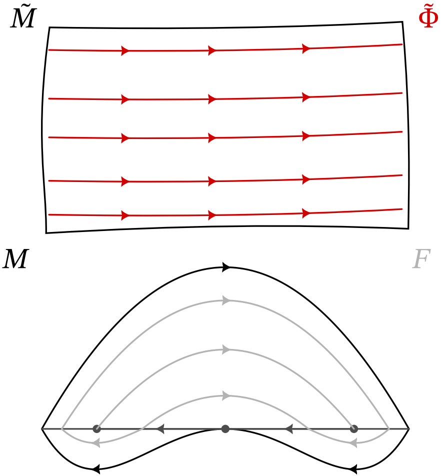

Orbit space and flow

Definition

The *orbit space* is

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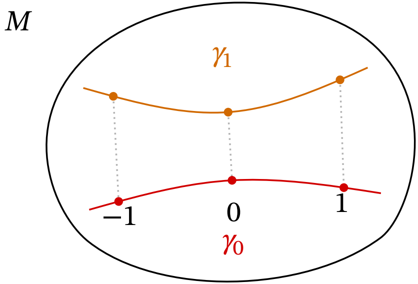
$$\tilde{M} := \{\gamma :\ \mathbb{R} \to M \text{ maximal orbit of } F\}.$$

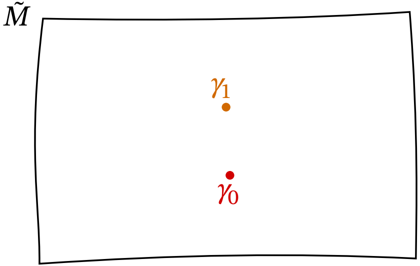
Definition

The *orbit space flow* is

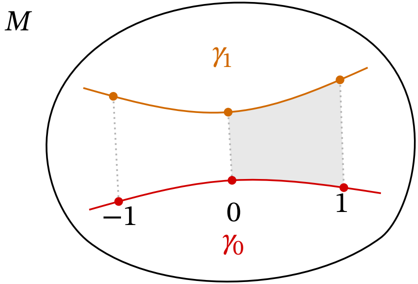
$$\begin{aligned} \tilde{\Phi} :\ \mathbb{R} \times \tilde{M} &\longrightarrow \tilde{M} \\ (t, \gamma) &\longmapsto \tilde{\Phi}^t(\gamma) :\ \mathbb{R} \longrightarrow M \\ &\qquad\qquad\qquad s \longmapsto \gamma(t+s). \end{aligned}$$

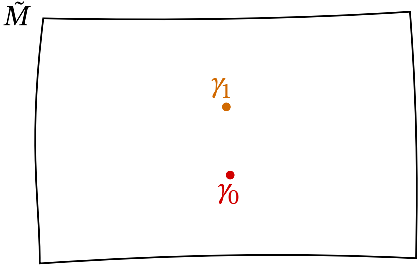
Distance on the orbit space



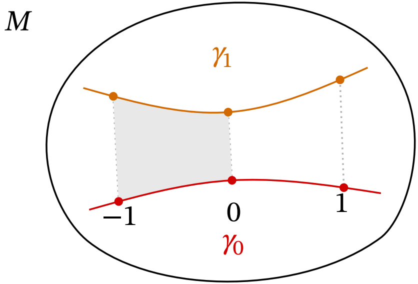


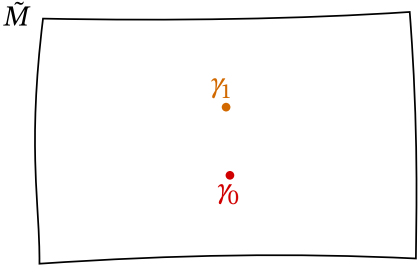
Distance on the orbit space



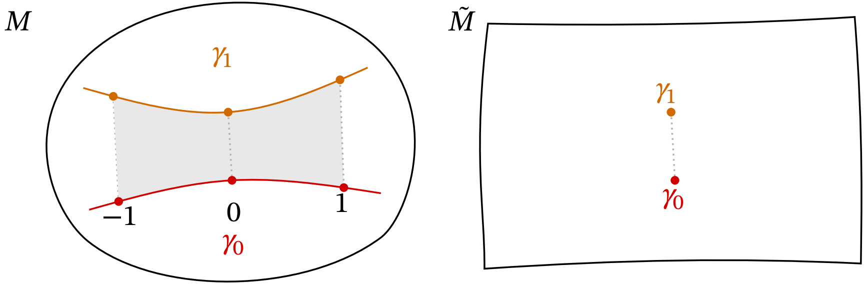


Distance on the orbit space





Distance on the orbit space



Distance on the orbit space

Definition

The *integral distance* on $\tilde{M}$ is

$$\tilde{d} : \tilde{M} \times \tilde{M} \to \mathbb{R}$$
$$\tilde{d}(\gamma_0, \gamma_1) := \sum_{i \in \mathbb{Z}} \frac{1}{2^{|i|}} \int_i^{i+1} d(\gamma_0(t), \gamma_1(t))\, \mathrm{d}t.$$

Topological properties

Theorem (Antunes, Carvalho, and Varão [1])

The flow $\tilde{\Phi}$ is continuous.

Theorem (Gomide, M—, Varão [7])

The orbit space $\tilde{M}$ is

- *separable*
- *Baire*
- *perfect (has no isolated points)*

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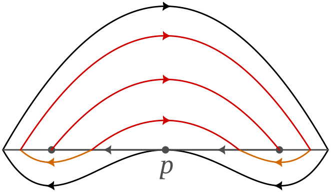
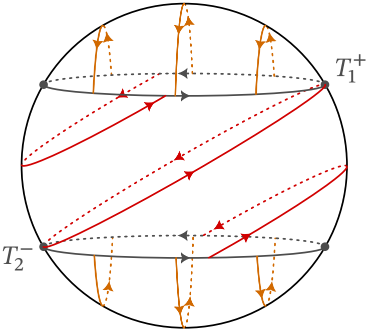
Theorem (Gomide, M—, Varão [7])

$\tilde{\Phi}$ is topologically transitive $\Leftrightarrow$ there exists a dense orbit.

Tangent points are connected

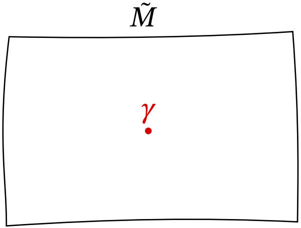


(1) Bean model

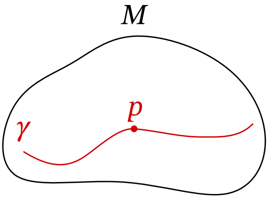


(2) Sphere model

Transitivity above and below

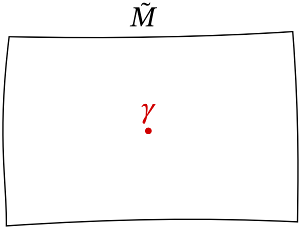


$\tilde{\Phi}$ is transitive



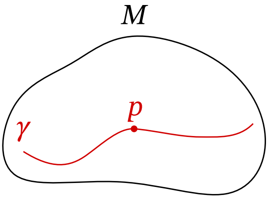
F is transitive

Transitivity above and below



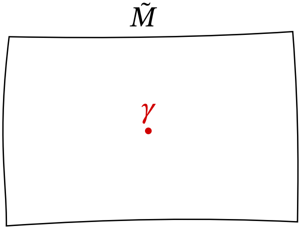
$\tilde{\Phi}$ is transitive

$\Downarrow$



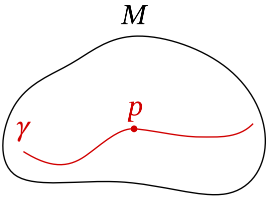
F is transitive

Transitivity above and below



$\tilde{\Phi}$ is transitive

$\Downarrow$ $\Uparrow$?



F is transitive

Transitivity above and below

Theorem (Gomide, M—, Varão [7])

$\tilde{\Phi}$ *is transitive* $\Rightarrow$ F *is transitive.*

Transitivity above and below

Theorem (Gomide, M—, Varão [7])

$\tilde{\Phi}$ is transitive $\Rightarrow$ F is transitive.

Theorem (Gomide, M—, Varão [7])

Suppose Σ^{t} is finite and there is $\mathcal{T} \subseteq \Sigma^{t}$ such that:

1. *Tangent points of $\mathcal{T}$ are connected;*
2. *Orbits reaches $\mathcal{T}$ on positive time;*
3. *Orbits reaches $\mathcal{T}$ on negative time.*

Then $\tilde{\Phi}$ is topologically transitive.

Section 4

Entropy of a PSVF

Topological entropy of PSVF

Definition (Antunes, Carvalho, and Varão [1])

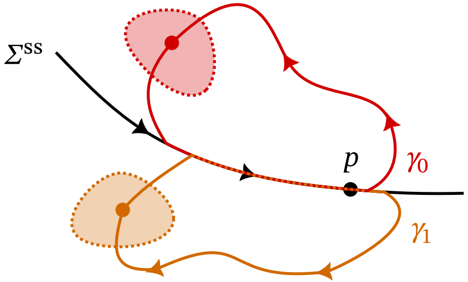
The *topological entropy* of F is the entropy of the orbit space flow $\tilde{\Phi}$:

$$\mathrm{h}(F) \coloneqq \mathrm{h}(\tilde{\Phi}^1).$$

Entropy estimation

Theorem (Euzébio, M—, Varão [6])

Let F be a 2-dimensional Filippov PSVF with $\Sigma^s \neq \emptyset$.
If F is transitive, then $\mathrm{h}(F) > 0$.



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[1] A. A. Antunes, T. Carvalho, and R. Varão. "On Topological Entropy of Piecewise Smooth Vector Fields". *Journal of Differential Equations* 362 (2023), pp. 52–73.

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[3] C. A. Buzzi, T. de Carvalho, and R. D. Euzébio. "Chaotic Planar Piecewise Smooth Vector Fields with Non-Trivial Minimal Sets". *Ergodic Theory and Dynamical Systems* 36.2 (2016), pp. 458–469.

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[5] R. D. Euzébio, J. S. Jucá, and R. Varão. "There Exist Transitive Piecewise Smooth Vector Fields on $\mathbb{S}^2$ but not Robustly Transitive". *Journal of Nonlinear Science* 32.4 (2022), No. 55.

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[6] R. D. Euzébio, P. G. Mattos, and R. Varão. "Characterization of Non-Deterministic Chaos in Two-Dimensional Non-Smooth Vector Fields". 2024.

[7] O. M. L. Gomide, P. G. Mattos, and R. Varão. "The Orbit Space Approach for Piecewise Smooth Vector Fields". 2023.