Entropy and Non-Deterministic Chaos for Piecewise Smooth Vector Fields

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1 Abstract

Our context is piecewise smooth vector fields (PSVF) defined on two-dimensional manifolds having a finite number of tangency points. We prove that topological transitivity is a necessary and sufficient condition for the occurrence of non-deterministic chaos when the PSVF system has non-empty sliding or escaping regions.

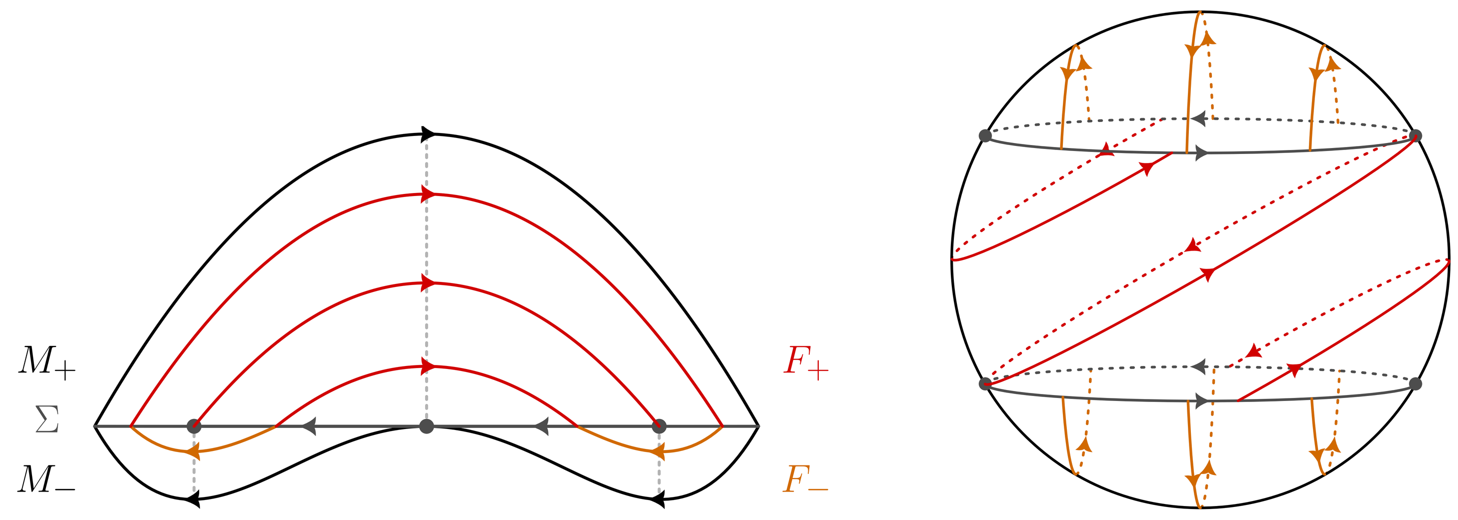
A fundamental result for continuous flows is the equivalence of topological transitivity and existence of a dense orbit. We prove in our setting that topological transitivity for PSVF systems is indeed equivalent to the existence of a dense orbit, although, in contrast to the continuous case, we are not able to garantee that the dense orbit implies the existence of a residual set of dense orbits.

Finally we prove that, in this context, topological transitivity implies strictly positive topological entropy for the PSVF system. This calculation is made using techniques similar to those from symbolic dynamics.

2 Preliminaries

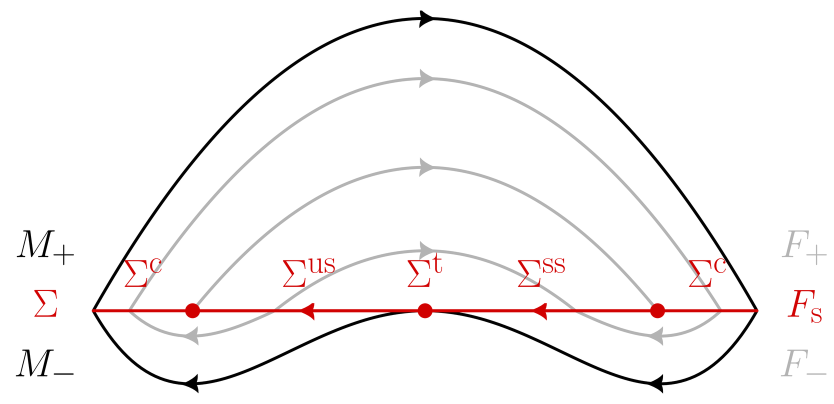
2.1 Piecewise smooth vector fields

Our setting is a (2-dimensional) Riemannian manifold M, partitioned into open submanifolds M_+, M_-, with smooth vector fields F_+ on M_+ and F_- on M_-. The boundary between M_+ and M_- is the **switching manifold** Σ.



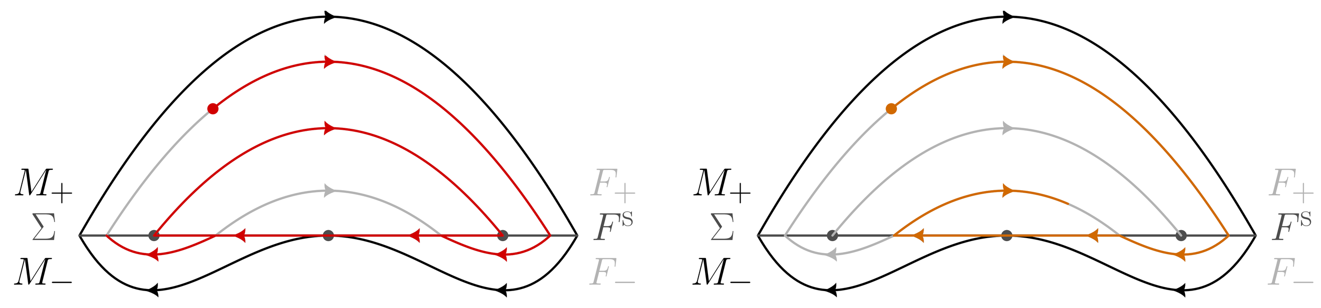
2.2 Switching regions and the sliding vector field

Filippov's convention induces the **sliding vector field** F_s on the sliding region Σ^s of Σ. The field F_s on each point of Σ^s is the unique convex combination of F_+ and F_- that is tangent to Σ^s.



2.3 PSVF orbits

PSVF orbits are concatenations of orbit segments of F_+, F_- and F_s.



3 Main results

Theorem 3.1 ([1]). *Assume that M is a two-dimensional manifold with a PSVF system having a finite number of tangency points. Then the Filippov system is topologically transitive on M if, and only if, there exists a dense PSVF orbit.*

Theorem 3.2 ([1]). *Let M be a two-dimensional manifold and F a transitive PSVF system having a finite number of tangency points. If the sliding region is non-empty, then the following statements hold:*

1. There is a dense set Δ such that, for every $x \in \Delta$,

1.1. there is a dense orbit through x;

1.2. the periodic orbits through x form a dense set;

2. The system has sensitive dependence on initial conditions.

Theorem 3.3 ([1]). *Assume that M is a two-dimensional manifold with a transitive PSVF system. If the sliding or escaping regions are non-empty, then the system has positive topological entropy.*

4 Transitivity implies chaos for PSVFs

Definition 1. Let F be a PSVF system on a manifold M. We define the following:

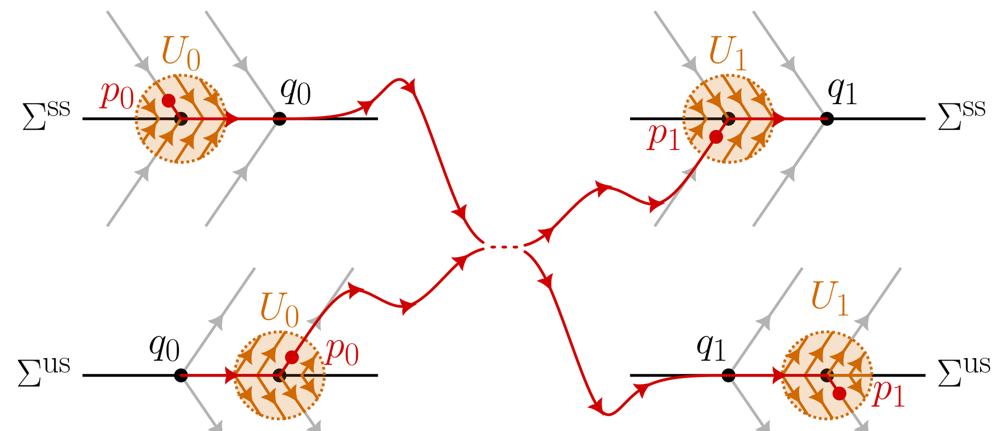
1. F is **topologically transitive** if given any two non-empty open sets U and V of M, there exists a PSVF orbit from a point of U to a point of V.
2. F exhibits **sensitive dependence on initial conditions** if there exists a fixed $\delta > 0$ such that, for any non-empty open set U, there exist points $x, y \in U$, PSVF orbits γ_x and γ_y which start at x and y, respectively, and some time t such that $d(\gamma_x(t), \gamma_y(t)) > \delta$.
3. a PSVF orbit γ is **periodic** if there is $\tau \in \mathbb{R}$ such that $\gamma(t) = \gamma(t+\tau)$ for every $t \in \mathbb{R}$.

We adopt a definition of chaos based on Devaney's.

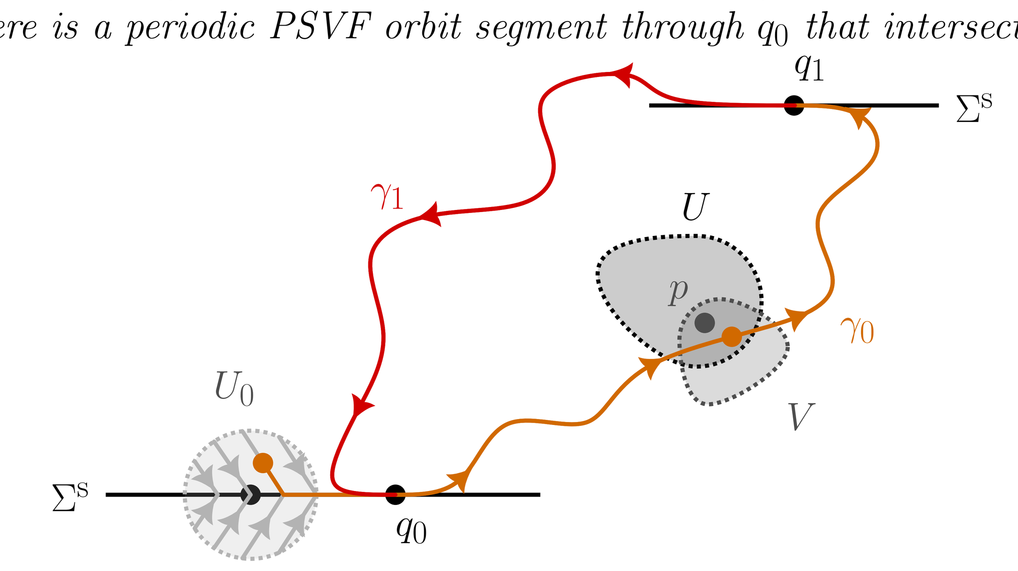
Definition 2 (Chaotic PSVFs). F is **chaotic** when

1. F is topologically transitive
2. F is sensitive dependence on initial conditions
3. The union of all PSVF periodic orbits is a dense set

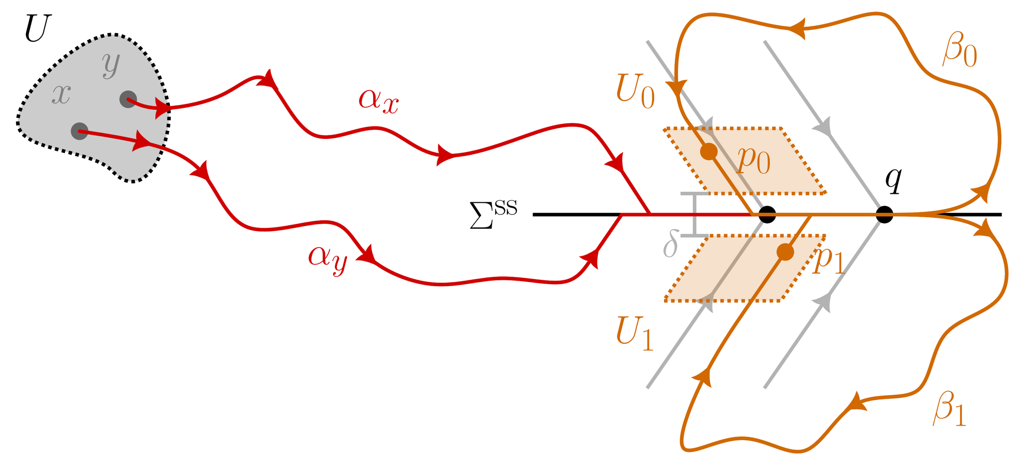
Lemma 4.1. *Assume topological transitivity. For every pair of points q_0, $q_1 \in \Sigma^s$, there is a PSVF orbit segment from q_0 to q_1.*



Lemma 4.2. *Assume topological transitivity and $\Sigma^s \neq \emptyset$. Let $q_0 \in \Sigma^s$ and $U \subseteq M$ be a non-empty open set. Then there is a periodic PSVF orbit segment through q_0 that intersects U.*



Lemma 4.3. *Assume topological transitivity. F has sensitive dependence on initial conditions.*



5 Topological entropy

5.1 Orbit spaces

Definition 3 (Orbit space).

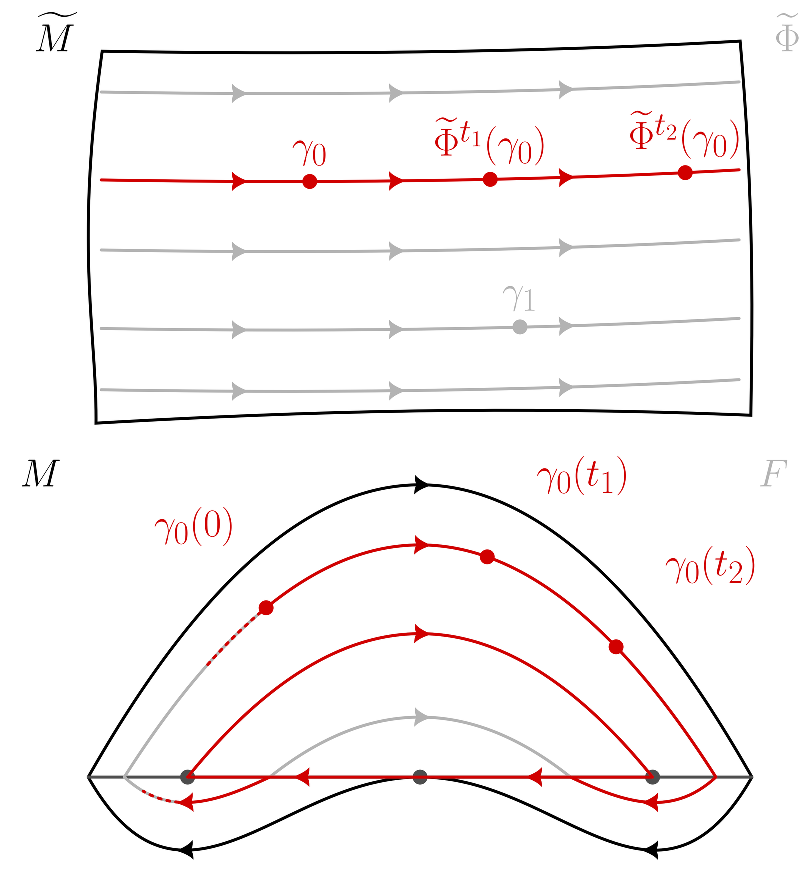
$$\widetilde{M} := \{\text{All maximal PSVF orbits of } F\}.$$

Definition 4 (Orbit space distance).

$$\widetilde{d}_{\sup}(\gamma_0, \gamma_1) := \sum_{i \in \mathbb{Z}} \frac{1}{2^{|i|}} \sup_{i \leq t < i+1} d(\gamma_0(t), \gamma_1(t)).$$

Definition 5 (Orbit space flow).

$$\begin{aligned} \widetilde{\Phi}^t \colon \widetilde{M} &\longrightarrow \widetilde{M} \\ \gamma &\longmapsto \widetilde{\Phi}^t(\gamma) \colon \mathbb{R} \longrightarrow M \\ & \qquad\qquad s \longmapsto \gamma(t+s). \end{aligned}$$

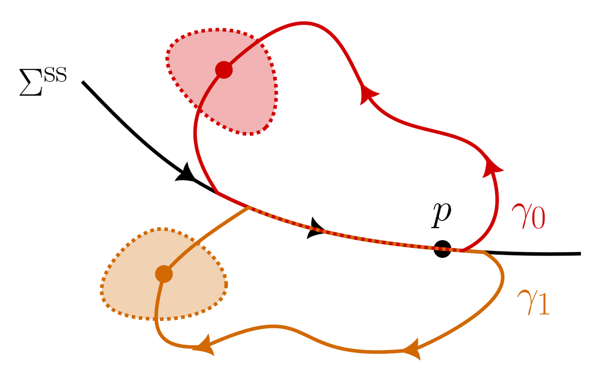


5.2 Topological entropy PSVFs

Definition 6. The **topological entropy** of F is the topological entropy of the time 1 map $\widetilde{\Phi}^1 : \widetilde{M} \to \widetilde{M}$ on the orbit space $\widetilde{M}$, denoted

$$h(F) := h(\widetilde{\Phi}^1).$$

Lemma 5.1. *Assume topological transitivity and $\Sigma^s \neq \emptyset$. There exist 2 distinct periodic PSVF orbit segments $\gamma_0, \gamma_1 : [0, \alpha] \to M$ of F with period $\alpha > 0$ and initial point $p = \gamma_0(0) = \gamma_0(\alpha) = \gamma_1(0) = \gamma_1(\alpha) \in M$.*



Proposition 5.2. $h(\widetilde{\Phi}^\alpha|_{\widetilde{\Gamma}_p}) \geq \log(2)$.

Proof. The proof uses techniques very similar the calculation of topological entropy in symbolic dynamics. □

References

[1] Rodrigo D. Euzébio, Pedro G. Mattos, and Régis Varão. **Characterization of Non-Deterministic Chaos in Two-Dimensional Non-Smooth Vector Fields**. 2024. arXiv: 2101.12025 [math.DS].

[2] Otávio M. L. Gomide, Pedro G. Mattos, and Régis Varão. **The Orbit Space Approach for Piecewise Smooth Vector Fields**. 2023. arXiv: 2310.13804 [math.DS].

Acknowledgement

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